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Rita A. Hibschweiler
Thomas H. MacGregor

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List of Symbols

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$\mathcal{D}$	the open unit disk	1
$\vert$	the set of complex numbers	1
T	the unit circle	1
$\mathcal{M}$	the set of complex-valued Borel measures on T	1
$\mathcal{M}^*$	the set of probability measures on T	1
$\mathcal{F}_\alpha$	the set of fractional Cauchy transforms of order α	1
$\Vert\mu\Vert$	the total variation of the measure μ	3
H^p	the Hardy space	3
$\Vert f\Vert_{H^p}$	the norm of f in H^p	3
$F(\theta)$	the radial limit of f in the direction $e^{i\theta}$	3
$f(e^{i\theta})$	the radial limit of f in the direction $e^{i\theta}$	3
$\mathcal{P}$	the set of normalized analytic functions with positive real part	4
F_α^*	the set of functions in $\mathcal{F}_\alpha$ represented by a probability measure	5
h^p	the set of harmonic functions in $\mathcal{D}$ with bounded integral means	5
$A_n(\alpha)$	the binomial coefficients	6
$\Vert f\Vert_{F_\alpha}$	the norm of f in $\mathcal{F}_\alpha$	11
H	the space of functions analytic in $\mathcal{D}$	13

Chapter 2.

S	the function $S(z) = \exp\left\{-\frac{1+z}{1-z}\right\}$	17
Γ	the gamma function	17
$\mathcal{C}$	the space of continuous functions on T	23
$\Vert f\Vert_C$	the norm of f in $\mathcal{C}$	23
$\mathcal{B}_\alpha$	the Besov space of order α	30
$\Vert f\Vert_{B_\alpha}$	the norm of f in $\mathcal{B}_\alpha$	30
$f * g$	the Hadamard product of f and g	41
$\mathcal{A}$	the Banach space of functions analytic in $\mathcal{D}$ and continuous in $\overline{D}$	44

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Preface

This book is an introduction to research on fractional Cauchy transforms. We study families of functions denoted $\mathcal{F}_\alpha$ where $\alpha \geq 0$. Functions in these families are analytic in the open unit disk, and $\mathcal{F}_\alpha$ is a Banach space. A function in one family corresponds to a function in another family through fractional differentiation or through fractional integration. When $\alpha = 1$ the family consists of the Cauchy transforms of complex-valued measures supported on the unit circle.

Our subject has its roots in classical complex analysis. The focus began with the research on Cauchy transforms initiated by V.P. Havin in 1958. Contributors to that development include A.B. Aleksandrov, M.G. Goluzina, V.P. Havin, S.V. Hrušcěv and S.A. Vinogradov. Research on the more general families of fractional Cauchy transforms started in the late 1980's. Mathematicians who have contributed to this area include J.A. Cima, D.J. Hallenbeck, R.A. Hibschweiler, T.H. MacGregor, E.A. Nordgren and K. Samotij.

Most of the work on the families $\mathcal{F}_\alpha$ took place in the 1990's. We present much of this work and several more recent results. We also give some of the earlier work on Cauchy transforms. A number of techniques and basic questions trace their roots back to this initial work. The forthcoming book "Cauchy Transforms on the Unit Circle" by J.A. Cima, A.L. Matheson and W.T. Ross gives a more extensive and up-to-date treatment of Cauchy transforms, including topics which we do not discuss.

A primary focus of this work is on concrete analytic questions about the structure of the family $\mathcal{F}_\alpha$ and about the properties of functions in $\mathcal{F}_\alpha$. The arguments given here make use of a variety of techniques from complex analysis and harmonic analysis, and often depend on a number of computations and technical facts. Functional analysis provides the general framework for this study. We frequently use such results as the Banach-Alaoglu theorem, the closed graph theorem, the Riesz representation theorem and the Hahn-Banach theorem.

The book begins with a survey of preliminary facts about the families $\mathcal{F}_\alpha$. We present classical results which are associated with complex-valued measures on the unit circle. This forms a base for the further development of our study. The initial facts about $\mathcal{F}_\alpha$ include formulas for mappings between the families, examples about infinite Blaschke products, and a result about Hadamard products and $\mathcal{F}_1$. The first significant result about membership of an analytic function in $\mathcal{F}_\alpha$ is a consequence of membership in a related Besov space.

Chapter 3 contains estimates on the integral means of functions in $\mathcal{F}_\alpha$. In particular, this provides a useful connection with the Hardy spaces H^p. We introduce the Dirichlet spaces and relate the membership of a function in the family $\mathcal{F}_\alpha$ with membership in the Dirichlet spaces and in the Besov spaces. In the case of an inner function, we give a definitive statement about membership in these three spaces.

Chapter 4 is a study of the radial and nontangential limits of functions in $\mathcal{F}_\alpha$. These limits and related facts are typically associated with various kinds of exceptional sets, including those having zero α-capacity. In Chapter 5 we consider the problem of describing the zeros of functions in $\mathcal{F}_\alpha$. One result yields a characterization of the zeros in the case $\alpha > 1$.

Chapters 6 and 7 are devoted to a study of the set of multipliers of $\mathcal{F}_\alpha$, which is denoted $\mathcal{M}_\alpha$. This is a rich and extensive area. We discuss a number of properties that are necessary for a function to belong to $\mathcal{M}_\alpha$. For example, such a function must be bounded and must have uniformly bounded radial variations. We prove a fundamental sufficient condition for membership in $\mathcal{M}_\alpha$ in the case $0 < \alpha < 1$. Additional sufficient conditions are obtained in Chapter 7. One of these conditions applies when $\alpha = 1$ and depends on showing that a Toeplitz operator is bounded on H^∞. Some of the sufficient conditions concern the smoothness of the boundary values of the function, and others are related to the Taylor coefficients. We also study the question of when an inner function belongs to $\mathcal{M}_\alpha$.

Chapter 8 concerns the composition of functions in $\mathcal{F}_\alpha$ with an analytic self-mapping of the unit disk. The main result asserts that when $\alpha \geq 1$, any self-mapping φ induces a composition operator on $\mathcal{F}_\alpha$. If φ is a conformal automorphism of the disk, then φ induces a composition operator on $\mathcal{F}_\alpha$ for all $\alpha > 0$. We use facts about composition to derive results about the factorization of functions in $\mathcal{F}_\alpha$ in terms of their zeros.

In Chapter 9 we discuss connections between the class of univalent functions and the families $\mathcal{F}_\alpha$. These connections are what first stimulated interest in $\mathcal{F}_\alpha$. In particular, we find that each analytic univalent function belongs to $\mathcal{F}_\alpha$ for $\alpha > 2$. Also we describe particular univalent functions that belong to $\mathcal{F}_2$. In the last chapter, we give an analytic characterization of the family of Cauchy transforms when considered as functions defined in the complement of the unit circle.

A number of open problems remain in this field. Some of these problems are mentioned in the text. Perhaps the most significant problem is to find an intrinsic analytic characterization of each family $\mathcal{F}_\alpha$.

We have tried to make our exposition as self-contained as possible. For example, the information we use about α-capacity is developed completely here. Likewise, our discussion of Toeplitz operators begins with a definition and yields a proof of the result needed in our application to $\mathcal{M}_1$. We include the proofs of nearly every result, including some well-known classical facts and some elementary technical facts. In general, the results we use about H^p or about the harmonic classes h^p are basic facts which can be found in various books in these areas. References are given for the few advanced results which we use without giving a proof.

The main background needed to read this book is an introduction to real analysis, complex analysis and functional analysis. Such a background is provided by the well-known books "Real Analysis" by H.L. Royden and "Complex Analysis" by L.V. Ahlfors. This book is suitable for advanced graduate courses and seminars. Preliminary forms of the book were developed in such settings.

Each chapter of the book opens with a preamble, which provides an overview of the development to follow. We end each chapter with a section called Notes, where we give references for the results in the chapter, as well as additional comments and references to related work. When appropriate, we provide a reference in the text precisely where a particular result is used. The statements in the book, including theorems, propositions, corollaries and lemmas, are numbered sequentially with the chapter number as a prefix. Thus Theorem 2.10 is the tenth statement appearing in Chapter 2. The end of a proof is indicated by the symbol □. We provide a list of special symbols which gives the page number on which each symbol first occurs. The reference list is given alphabetically by author and then by year. Thus Hallenbeck [1997] refers to a research paper by D.J. Hallenbeck published in 1997.

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Dedication

Rita Hibschweiler dedicates this book to the memory of her father, Warren, to her mother, Cecelia, and to her daughter, Jean.

Thomas MacGregor dedicates this book to the memory of his mother and father, Frances and John, and to his brother, Mickey.

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CHAPTER 1

Introduction

Preamble. This chapter introduces the topic of this book, the families of fractional Cauchy transforms. For each $\alpha \geq 0$ a family $\mathcal{F}_\alpha$ is defined as the collection of functions that can be expressed as the Cauchy-Stieltjes integral of a suitable kernel. The case $\alpha = 1$ corresponds to the set of Cauchy transforms of measures on the unit circle $T = \{z \in \mathbb{C} : |z| = 1\}$. Each function in $\mathcal{F}_\alpha$ is analytic in $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Some facts are recalled about the Hardy spaces H^p and the harmonic classes h^p, and it is noted that $H^1 \subset \mathcal{F}_1$. Other connections between H^p and $\mathcal{F}_\alpha$ are given in Chapter 3 and in later chapters. Properties of complex-valued measures on T are obtained. Subsequently, these properties are shown to be related to properties of functions in $\mathcal{F}_\alpha$.

The Riesz-Herglotz formula is quoted and the correspondence between measures and functions given by this formula is shown to be one-to-one. As a consequence of this formula, any function which is analytic in $\mathcal{D}$ and has a range contained in a half-plane belongs to the family $\mathcal{F}_1$.

The F. and M. Riesz theorem yields a description of all measures representing a particular function in $\mathcal{F}_\alpha$. A norm is defined on $\mathcal{F}_\alpha$ as the infimum of the total variation norms of the measures representing the function. For each function f in $\mathcal{F}_\alpha$, there is a representing measure having minimal norm. The family $\mathcal{F}_\alpha$ is a Banach space with respect to the given norm, and convergence in the norm of $\mathcal{F}_\alpha$ implies convergence which is uniform on compact subsets of $\mathcal{D}$.

Let $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $T = \{z \in \mathbb{C} : |z| = 1\}$. Let $\mathcal{M}$ denote the set of complex-valued Borel measures on T and let $\mathcal{M}^*$ denote the subset of $\mathcal{M}$ consisting of probability measures. For each $\alpha > 0$ we define $\mathcal{F}_\alpha$ in the following way: $f \in \mathcal{F}_\alpha$ provided that there exists $\mu \in \mathcal{M}$ such that

$$f(z) = \int_T \frac{1}{(1 - \overline{\zeta} z)^\alpha} \, d\mu(\zeta) \qquad (|z| < 1). \tag{1.1}$$

We also define $\mathcal{F}_0$ as consisting of functions f such that

$$f(z) = f(0) + \int_T \log \frac{1}{1-\bar{\zeta}z}\, d\mu(\zeta) \qquad (|z|<1) \tag{1.2}$$

where $\mu \in \mathcal{M}$. Throughout we use the principal branch of the logarithm and the power functions. Each function given by (1.1) or by (1.2) is analytic in $\mathcal{D}$ and the derivatives of f can be obtained by differentiation of the integrand with respect to z.

The family $\mathcal{F}_\alpha$ is a vector space over $\mathbb{C}$ with respect to the usual addition of functions and multiplication of functions by complex numbers. The family $\mathcal{F}_1$ is of special importance because of its connection with the Cauchy formula. When $\alpha = 1$, (1.1) can be rewritten

$$f(z) = \int_T \frac{1}{\zeta - z}\, d\upsilon(\zeta) \qquad (|z|<1) \tag{1.3}$$

where $d\upsilon(\zeta) = \zeta\, d\mu(\zeta)$. The correspondence $\mu \to \upsilon$ just described gives a one-to-one mapping of $\mathcal{M}$ onto $\mathcal{M}$. Hence, when $\alpha = 1$ the set of functions given by (1.1) is the same as the set of functions given by (1.3), where μ and υ vary in $\mathcal{M}$.

Suppose that the function f is analytic in $\bar{D}$. Then the Cauchy formula gives

$$f(z) = \frac{1}{2\pi i}\int_T \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad (|z|<1). \tag{1.4}$$

Hence f has the form (1.3) where $d\upsilon(\zeta) = f(\zeta)\, d\zeta / 2\pi i$. The formula (1.4) holds more generally when f belongs to the Hardy space H^1 and where $f(\zeta)$ is defined almost everywhere on T by $f(\zeta) = \lim_{r\to 1-} f(r\zeta)$. Hence $H^1 \subset \mathcal{F}_1$.

Conversely, suppose that $f(z) = \int_T \frac{1}{1-\bar{\zeta}z}\, d\mu(\zeta)$, $\mu \in \mathcal{M}$ and $\int_T \zeta^n\, d\mu(\zeta) = 0$ for n = 1,2,3,… . As we now show this implies that $f \in H^1$. We have

$$f(z) = \int_T \sum_{n=0}^{\infty} (\bar{\zeta}z)^n\, d\mu(\zeta)$$

$$= \int_T d\mu(\zeta) + \sum_{n=1}^{\infty} \{\int_T \bar{\zeta}^n z^n \, d\mu(\zeta) + \int_T \zeta^n \bar{z}^n \, d\mu(\zeta)\}$$

$$= \int_T \{1 + \sum_{n=1}^{\infty} 2 \operatorname{Re} (\bar{\zeta} z)^n\} \, d\mu(\zeta)$$

$$= \int_T \operatorname{Re} \left\{\frac{1 + \bar{\zeta} z}{1 - \bar{\zeta} z}\right\} d\mu(\zeta).$$

By the mean-value property of harmonic functions we see that if $0 < r < 1$, then

$$\int_{-\pi}^{\pi} | f(re^{i\theta}) | \, d\theta \le \int_T \int_{-\pi}^{\pi} \operatorname{Re} \left\{\frac{1 + \bar{\zeta} re^{i\theta}}{1 - \bar{\zeta} re^{i\theta}}\right\} d\theta \, d|\mu|(\zeta)$$
$$= 2\pi \, \|\mu\| < \infty,$$

where $\|\mu\|$ denotes the total variation of the measure μ. Therefore f 0 H^1.

We recall a few facts about the Hardy spaces H^p. For $0 < p < \infty$, H^p is defined in the following way: f 0 H^p provided that f is analytic in $\mathbb{D}$ and

$$\sup_{0<r<1} \int_{-\pi}^{\pi} | f(re^{i\theta}) |^p \, d\theta < \infty. \tag{1.5}$$

When $p = \infty$, H^p is the set of functions that are analytic and bounded in $\mathbb{D}$. For $0 < p \le \infty$, H^p is a vector space and for $1 \le p \le \infty$, H^p is a Banach space, where the norm is defined by

$$\| f \|_{H^p} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} | f(re^{i\theta}) |^p \, d\theta\right)^{1/p} \tag{1.6}$$

for $p < \infty$ and

$$\| f \|_{H^\infty} = \sup_{|z|<1} | f(z) |. \tag{1.7}$$

If f 0 H^p for some $p > 0$ then $F(\theta) \equiv \lim_{r\to 1-} f(r\zeta)$ exists for almost all θ in $[-\pi, \pi]$, where $\zeta = e^{i\theta}$ and F 0 L^p ($[-\pi, \pi]$). We also use the notation $f(\zeta) = f(e^{i\theta})$ for this limit.

The Riesz-Herglotz formula is related to our study, and is given in the following theorem. Let $\mathcal{P}$ denote the set of functions f such that f is analytic in $\mathcal{D}$, $f(0) = 1$, and Re $f(z) > 0$ for $|z| < 1$.

Theorem 1.1 *f 0 $\mathcal{P}$ if and only if there exists* μ 0 $\mathcal{M}^*$ *such that*

$$f(z) = \int_T \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta) \qquad (|z| < 1). \tag{1.8}$$

Theorem 1.1 implies that $\mathcal{P}$ δ $\mathcal{F}_1$. To see this, suppose that f is given by (1.8) and μ 0 $\mathcal{M}^*$. Let λ 0 $\mathcal{M}$ be defined by $d\lambda(\zeta) = (1/2\pi i\zeta)\, d\zeta$. Then

$$\int_T \overline{\zeta}^n\, d\lambda(\zeta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\, d\theta$$

$$= \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n = 1,2,\ldots . \end{cases}$$

Hence

$$\int_T \frac{1}{1 - \overline{\zeta}z}\, d\lambda(\zeta) = 1 \qquad (|z| < 1).$$

Thus (1.8) can be rewritten

$$f(z) = \int_T \frac{1}{1 - \overline{\zeta}z}\, d\upsilon(\zeta),$$

where $\upsilon = 2\mu - \lambda$.

More generally, if the function f is analytic in $\mathcal{D}$ and $f(\mathcal{D})$ is contained in some half-plane, it easily follows that f 0 $\mathcal{F}_1$.

Suppose that μ 0 $\mathcal{M}$. Then the Jordan decomposition theorem implies that

$$\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4 \tag{1.9}$$

where each μ_n is a nonnegative measure on T. A simple argument using the Hahn decomposition shows that

$$\sum_{k=1}^{4} \mu_k \;(T) \le \sqrt{2}\;\|\mu\|. \tag{1.10}$$

We have $\mu_n = a_n \upsilon_n$ where $a_n \ge 0$ and υ_n 0 $\mathcal{M}^*$. We let $\mathcal{F}_\alpha^*$ denote the subset of $\mathcal{F}_\alpha$ consisting of functions which can be represented by some μ 0 $\mathcal{M}^*$. Then each f 0 $\mathcal{F}_\alpha$ can be written

$$f = a_1 f_1 - a_2 f_2 + ia_3 f_3 - ia_4 f_4 \tag{1.11}$$

where $a_n \ge 0$ and f_n 0 $\mathcal{F}_\alpha^*$ for n = 1,2,3,4.

Functions in $\mathcal{F}_1$ which can be represented by real measures are related to the class h^1. We recall that h^p $(0 < p < \infty)$ is defined as the set of complex-valued functions u that are harmonic in $\mathbb{D}$ and satisfy

$$\sup_{0<r<1} \int_{-\pi}^{\pi} |u(re^{i\theta})|^p \, d\theta < \infty.$$

A function u belongs to h^1 if and only if u has a Poisson-Stieltjes representation

$$u(z) = \frac{1}{2\pi} \int_T \operatorname{Re}\left\{\frac{\zeta + z}{\zeta - z}\right\} d\mu(\zeta) \qquad (|z| < 1) \tag{1.12}$$

where μ 0 $\mathcal{M}$.

Suppose that the function f is analytic in $\mathbb{D}$ and u = Re (f) 0 h^1. Then (1.12) holds where μ is a real-valued measure. This implies that

$$f(z) = \frac{1}{2\pi} \int_T \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta) + i \,\operatorname{Im} f(0) \tag{1.13}$$

for $|z| < 1$ and hence f 0 $\mathcal{F}_1$. If, in addition, $f(0)$ is real then f can be represented in $\mathcal{F}_1$ using a real measure. The converse also holds, that is, if f 0 $\mathcal{F}_1$ can be represented by a real measure then Re (f) 0 h^1 and $f(0)$ is real.

Suppose that μ 0 $\mathcal{M}$. Then μ can be identified with a measure υ defined on $(-\pi, \pi]$. The measure υ can be extended to $[-\pi, \pi]$ by letting $\upsilon(\{-\pi\}) = 0$. A complex-valued function g of bounded variation on $[-\pi, \pi]$ can be associated with υ in the following way. First apply the Jordan decomposition theorem to υ and then associate a nondecreasing function with each nonnegative part in this decomposition. When $\upsilon \geq 0$ on $[-\pi, \pi]$ the nondecreasing function, say *h*, is defined by $h(\mathrm{t}) = \upsilon([-\pi, \mathrm{t}])$ for $-\pi < \mathrm{t} \leq \pi$ and $h(-\pi) = 0$. Conversely each nondecreasing function h determines a nonnegative measure by first letting $\upsilon([\mathrm{a}, \mathrm{b})) = h(\mathrm{b}) - h(\mathrm{a})$, where $-\pi \leq \mathrm{a} < \mathrm{b} \leq \pi$, and then extending υ to the Borel subsets of $[-\pi, \pi]$. The correspondence $\upsilon \to g$ is one-to-one if we require the normalization that $g(\mathrm{t}) = \frac{1}{2}[g(\mathrm{t}-) + g(\mathrm{t}+)]$. Hence each μ 0 $\mathcal{M}$ yields a complex-valued function g of bounded variation in $[-\pi, \pi]$ and the formula (1.1) can be rewritten as

$$f(\mathrm{z}) = \int_{-\pi}^{\pi} \frac{1}{(1 - e^{-it}z)^{\alpha}} \, dg(t) \qquad (|z| < 1). \qquad (1.14)$$

For α 0 | and n a nonnegative integer $A_n(\alpha)$ is defined by the power series development

$$\frac{1}{(1 - z)^{\alpha}} = \sum_{n=0}^{\infty} A_n(\alpha) \, z^n \qquad (|z| < 1). \qquad (1.15)$$

The Taylor series for the binomial series gives

$$A_n(\alpha) = \frac{\alpha(\alpha + 1) \cdots (\alpha + n - 1)}{n!}. \qquad (1.16)$$

Suppose that f is analytic in $\mathcal{D}$ and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < 1). \qquad (1.17)$$

By comparing the coefficients of the power series for both sides of (1.1) we find that f 0 $\mathcal{F}_{\alpha}$ if and only if there exists μ 0 $\mathcal{M}$ such that

$$a_n = A_n(\alpha) \int_{T} \overline{\zeta}^{n} \, d\mu(\zeta) \qquad (1.18)$$

for n = 0, 1, … . Let f be given by (1.17) and let $g(z) = \sum_{n=0}^{\infty} b_n z^n$ $(|z| < 1)$ where $b_n = a_n / A_n(\alpha)$. The relation (1.18) implies that f 0 $\mathcal{F}_{\alpha}$ if and only if g 0 $\mathcal{F}_1$.

We shall give a description of the set of measures which represent a particular function in $\mathcal{F}_\alpha$. A critical step in the argument uses the following result of F. and M. Riesz.

Theorem 1.2 If μ 0 $\mathcal{M}$ *and* $\int_T \zeta^n \, d\mu(\zeta) = 0$ *for* n = 0, 1, … *then* μ *is absolutely continuous with respect to Lebesgue measure.*

Suppose that $\alpha > 0$, f 0 $\mathcal{F}_\alpha$ and (1.1) holds where μ 0 $\mathcal{M}$. Let g 0 H^1 and $g(0) = 0$. Cauchy's theorem and $g(0) = 0$ imply that if $0 < r < 1$ then

$$\int_{|z|=r} z^n \, g(z)\, dz = 0 \text{ for } n = -1, 0, 1, \ldots .$$

Because g 0 H^1 it follows that

$$\lim_{r \to 1-} \int_T \zeta^n \, g(r\zeta)\, d\zeta = \int_T \zeta^n \, g(\zeta)\, d\zeta \text{ for } n = -1, 0, 1, \ldots .$$

Therefore

$$\int_{-\pi}^{\pi} e^{in\theta} \, g(e^{i\theta})\, d\theta = 0 \qquad (1.19)$$

for n = 0, 1, … .

We claim that the measure υ where

$$d\upsilon(\zeta) = d\mu(\zeta) + \overline{g(e^{i\theta})}\, d\theta \qquad (1.20)$$

also represents f. It suffices to show that the function h defined by

$$h(z) = \int_T \frac{\overline{g(\zeta)}}{(1 - \overline{\zeta} z)^\alpha} \, \overline{\zeta}\, d\zeta \qquad (|z| < 1) \qquad (1.21)$$

is the zero function. Note that if $h(z) = \sum_{n=0}^{\infty} b_n z^n \;\; (|z| < 1)$ then

$$b_n = i\, A_n(\alpha) \overline{\int_{-\pi}^{\pi} e^{in\theta} \, g(e^{i\theta})\, d\theta}$$

for $n = 0, 1, \ldots$. Hence (1.19) implies $b_n = 0$ and so $h = 0$.

Next let $f \in \mathcal{F}_\alpha$ and suppose that f is represented by $\mu \in \mathcal{M}$. We show that if $\nu \in \mathcal{M}$ and ν represents f, then ν has the form (1.20) where $g \in H^1$ and $g(0) = 0$. Suppose that

$$f(z) = \int_T \frac{1}{(1-\overline{\zeta}z)^\alpha}\, d\nu(\zeta)$$

where $\nu \in \mathcal{M}$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(|z| < 1)$. Since (1.1) also holds, (1.18) and $A_n(\alpha) \neq 0$ imply

$$\int_T \overline{\zeta}^n\, d\mu(\zeta) = \int_T \overline{\zeta}^n\, d\nu(\zeta) \quad \text{for } n = 0, 1, 2, \ldots .$$

Let $\lambda = \nu - \mu$. Then

$$\int_T \overline{\zeta}^n\, d\lambda(\zeta) = 0 \quad \text{for } n = 0, 1, \ldots .$$

Define the measure σ by $\sigma(E) = \overline{\lambda(E)}$ for each Borel set $E \subset T$. Then

$$\int_T \zeta^n\, d\sigma(\zeta) = 0 \quad \text{for } n = 0, 1, \ldots . \tag{1.22}$$

Theorem 1.2 implies that $d\sigma(\zeta) = G(\theta)\, d\theta$ where $G \in L^1([-\pi, \pi])$ and $\zeta = e^{i\theta}$. Define the function g by

$$g(z) = \frac{1}{2\pi i}\int_T \frac{G(\theta)}{\zeta - z}\, d\zeta \qquad (|z| < 1). \tag{1.23}$$

Then g is analytic in $\mathbb{D}$ and $g(0) = \frac{1}{2\pi}\int_T d\sigma(\zeta) = 0$. From (1.22) we see that if

$|z| < 1$ then

$$\int_T \frac{\bar{\zeta}\, G(\theta)}{1-\zeta\bar{z}}\, d\zeta = \sum_{n=0}^{\infty} \left\{ \int_T \zeta^{n-1}\, G(\theta)\, d\zeta \right\} \bar{z}^n$$
$$= \sum_{n=0}^{\infty} \left\{ i \int_T \zeta^n\, d\sigma(\zeta) \right\} \bar{z}^n = 0.$$

Thus (1.23) gives

$$g(z) = \frac{1}{2\pi i} \int_T \frac{G(\theta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_T \frac{\bar{\zeta}\, G(\theta)}{1-\zeta\bar{z}}\, d\zeta - \frac{1}{2\pi i} \int_T \bar{\zeta}\, G(\theta)\, d\zeta.$$

Combining these integrals we obtain

$$g(z) = \frac{1}{2\pi i} \int_T \operatorname{Re} \left\{ \frac{1+\bar{\zeta} z}{1-\bar{\zeta} z} \right\} \bar{\zeta}\, G(\theta)\, d\zeta,$$

which exhibits g as the Poisson integral of G. Therefore $g \in H^1$ and $\lim_{r\to 1-} g(re^{i\theta}) = G(\theta)$ for almost all θ. We have shown that $\upsilon = \mu + \lambda$ where $d\lambda(\zeta) = \overline{g(e^{i\theta})}\, d\theta$, $g \in H^1$ and $g(0) = 0$.

The following statement summarizes what was just obtained. The argument given above also applies in the case $\alpha = 0$. Indeed, it applies to any class of functions defined by $\int_T F(\bar{\zeta} z)\, d\mu(\zeta)$ where $\mu \in \mathcal{M}$ and F is analytic in $\mathbb{D}$ and the Taylor coefficients for F are nonzero for n = 0,1,2,… .

Theorem 1.3 *Let $f \in \mathcal{F}_\alpha$. The set of measures in $\mathcal{M}$ representing f is given by the collection $\{\upsilon\}$ where $\upsilon = \mu + \lambda$, μ is any measure in $\mathcal{M}$ which represents f and λ varies over the measures described by $d\lambda(\zeta) = \overline{g(e^{i\theta})}\, d\theta$ where $g \in H^1$ and $g(0) = 0$.*

Theorem 1.4 *If $\mu \in \mathcal{M}$ and*

$$\int_T \bar{\zeta}^n\, d\mu(\zeta) = 0 \tag{1.24}$$

for all integers n, *then* μ *is the zero measure.*

Proof: The assumption (1.24) implies that

$$\int_T P(\zeta)\, d\mu(\zeta) = 0 \qquad (1.25)$$

for every trigonometric polynomial $P(\zeta) = \sum_{n=-m}^{m} a_n \zeta^n$. Hence the Weierstrass approximation theorem implies

$$\int_T F(\zeta)\, d\mu(\zeta) = 0 \qquad (1.26)$$

for all functions F continuous on T. Every function G integrable with respect to μ on T can be approximated in the norm $L^1(d\mu)$ by a continuous function. Hence

$$\int_T G(\zeta)\, d\mu(\zeta) = 0 \qquad (1.27)$$

for all functions G integrable with respect to μ. In particular, (1.27) holds where G is the characteristic function of an arc I on T. Thus μ(I) = 0 for every arc I on T. Therefore μ = 0.

Corollary 1.5 *If* μ *and* υ *are real-valued Borel measures in* $\mathcal{M}$ *and*

$$\int_T \overline{\zeta}^n\, d\mu(\zeta) = \int_T \overline{\zeta}^n\, d\upsilon(\zeta) \qquad (1.28)$$

for n = 0, 1, 2, … *then* μ = υ.

Proof: Since μ and υ are real-valued measures, by taking the conjugate of both sides of (1.28) we obtain

$$\int_T \zeta^n\, d\mu(\zeta) = \int_T \zeta^n\, d\upsilon(\zeta)$$

for n = 0, 1, 2, ... Thus (1.28) holds for all integers n. Hence λ = μ − υ satisfies the assumptions of Theorem 1.4. Therefore μ = υ.

Corollary 1.6 *The map* $\mu \mapsto f$ *from* $\mathcal{M}^*$ to $\mathcal{P}$ *given by* (1.8) *is one-to-one.*

Proof: Suppose that (1.8) holds where $\mu \in \mathcal{M}^*$ and also

$$f(z) = \int_T \frac{\zeta + z}{\zeta - z}\, d\upsilon(\zeta)$$

where $\upsilon \in \mathcal{M}^*$. If we let $f(z) = \sum_{n=0}^{\infty} a_n z^n \; (|z| < 1)$ then

$$a_n = 2\int_T \bar{\zeta}^n \, d\mu(\zeta) \text{ and } a_n = 2\int_T \bar{\zeta}^n \, d\upsilon(\zeta)$$

for n = 1, 2, … . Since

$$\int_T d\mu(\zeta) = 1 = \int_T d\upsilon(\zeta)$$

the hypotheses of Corollary 1.5 are satisfied. Therefore $\mu = \upsilon$.

Suppose that $f \in \mathcal{F}_\alpha$. If $\alpha > 0$ we let

$$\| f \|_{\mathcal{F}_\alpha} = \inf \| \mu \| \tag{1.29}$$

where μ varies over all measures in $\mathcal{M}$ for which (1.1) holds. In the case $\alpha = 0$ we let

$$\| f \|_{\mathcal{F}_0} = | f(0) | + \inf \| \mu \| \tag{1.30}$$

where μ varies over the measures for which (1.2) holds. We shall show that (1.29) and (1.30) define a norm on $\mathcal{F}_\alpha$ and with respect to this norm $\mathcal{F}_\alpha$ is a Banach space. We also show that the infimum in (1.29) is actually attained by some measure.

Lemma 1.7 *Suppose that* $\alpha > 0$ *and* $f \in \mathcal{F}_\alpha$. *Let* $\mathcal{M}_f$ *denote the set of measures in* $\mathcal{M}$ *that represent* f. *Then* $\mathcal{M}_f$ *is closed in the weak* topology.*

Proof: Suppose that $\alpha > 0$, $\mu_n \in \mathcal{M}_f$ for n = 1, 2, …, $\mu \in \mathcal{M}$ and $\mu_n \to \mu$ in the weak* topology. We have

$$f(z) = \int_T \frac{1}{(1-\bar{\zeta}z)^{\alpha}}\, d\mu_n(\zeta) \qquad (|z|<1) \qquad (1.31)$$

for n = 1, 2, … . For each z, (|z| < 1) the map $\zeta \mapsto (1-\bar{\zeta}z)^{-\alpha}$ is continuous on T. Since $\mu_n \to \mu$ this implies that

$$\int_T \frac{1}{(1-\bar{\zeta}z)^{\alpha}}\, d\mu_n(\zeta) \to \int_T \frac{1}{(1-\bar{\zeta}z)^{\alpha}}\, d\mu(\zeta) \qquad (1.32)$$

as $n \to \infty$. From (1.31) and (1.32) we get

$$f(z) = \int_T \frac{1}{(1-\bar{\zeta}z)^{\alpha}}\, d\mu(\zeta) \text{ for } |z|<1.$$

Hence $\mu \in \mathcal{M}_f$.

Theorem 1.8 *Suppose that* $\alpha > 0$ *and* $f \in \mathcal{F}_\alpha$. *Then there exists* $\upsilon \in \mathcal{M}_f$ *such that* $\|\upsilon\| = \|f\|_{\mathcal{F}_\alpha}$.

Proof: For each $R \geq 0$ let $\mathcal{M}(R) = \{\mu \in \mathcal{M} : \|\mu\| \leq R\}$ and let $\mathcal{M}_f(R) = \mathcal{M}_f \cap \mathcal{M}(R)$. Let $\mu_0 \in \mathcal{M}_f$ and set $R_0 = \|\mu_0\|$. The Banach-Alaoglu Theorem implies that $\mathcal{M}(R_0)$ is compact in the weak* topology. By Lemma 1.7, $\mathcal{M}_f$ is closed and thus $\mathcal{M}_f(R_0)$ is closed. Hence the compactness of $\mathcal{M}(R_0)$ implies that $\mathcal{M}_f(R_0)$ is compact.

Let $m = \|f\|_{\mathcal{F}_\alpha}$ and let

$$m' = \inf\{\|\mu\| : \mu \in \mathcal{M}_f(R_0)\}.$$

Clearly $m' = m$. There is a sequence $\{\mu_n\}$ in $\mathcal{M}_f(R_0)$ such that $\|\mu_n\| \to m$. The compactness of $\mathcal{M}_f(R_0)$ implies that there is a subsequence $\{\mu_{n_k}\}$ (k = 1, 2, ...) and $\upsilon \in \mathcal{M}_f(R_0)$ such that $\mu_{n_k} \to \upsilon$ as $k \to \infty$. Let $\varepsilon > 0$. Since $\|\mu_{n_k}\| \to m$ as $k \to \infty$ we have $\|\mu_{n_k}\| \leq m + \varepsilon$ for all sufficiently large values of k. Since $\mathcal{M}(m+\varepsilon)$ is compact, as above we see that $\mathcal{M}_f(m+\varepsilon)$ is compact. Therefore, $\upsilon \in \mathcal{M}(m+\varepsilon)$. We have $\|\upsilon\| \leq m + \varepsilon$ for every $\varepsilon > 0$. Thus $\|\upsilon\| \leq m$. Clearly $\|\upsilon\| \geq m$ and therefore $\|\upsilon\| = m$, as required. □

A result similar to Theorem 1.8 holds for $\alpha = 0$.

Suppose that (1.1) holds, μ 0 $\mathcal{M}$ and $\mu \geq 0$. Then $f(0) = \int_T d\mu(\zeta) = \| \mu \|$.

Suppose that υ 0 $\mathcal{M}$ and υ also represents f. Then

$$\| \mu \| = f(0) = | \int_T d\upsilon(\zeta) | \leq \| \upsilon \| .$$

This shows that $\| f \|_{\mathcal{F}_\alpha} = \| \mu \|$. In particular if μ 0 $\mathcal{M}^*$ and μ represents f in $\mathcal{F}_\alpha$ then $\| f \|_{\mathcal{F}_\alpha} = 1$. For example, if $| \zeta | = 1$ and $f(z) = 1/(1 - \overline{\zeta} z)^\alpha$ ($|z| < 1$) then f 0 $\mathcal{F}_\alpha$ and f can be represented by unit mass at ζ. Therefore $\| 1/(1 - \overline{\zeta} z)^\alpha \|_{\mathcal{F}_\alpha} = 1$.

We next show that (1.29) defines a norm on $\mathcal{F}_\alpha$ for $\alpha > 0$ and with respect to that norm $\mathcal{F}_\alpha$ is a Banach space. The argument for (1.30) is similar. The mapping from $\mathcal{M}$ to $\mathcal{F}_\alpha$ defined by (1.1) is linear. The kernel $\mathcal{K}$ of this mapping consists of the measures μ 0 $\mathcal{M}$ such that $\int_T \overline{\zeta}^n \, d\mu(\zeta) = 0$ for n = 0, 1, … . Because $\mathcal{M}$ is a Banach space with respect to the total variation norm and $\mathcal{K}$ is closed by Lemma 1.7, the collection of cosets $\mathcal{M}$э$\mathcal{K}$, with the usual addition of cosets and multiplication of cosets by scalars, also is a Banach space. The norm defined on $\mathcal{M}$э$\mathcal{K}$ is simply transferred to a norm on $\mathcal{F}_\alpha$ as given by (1.29). The exact information about $\mathcal{K}$ given in Theorem 1.3 is not needed to see these general relations.

Let H denote the topological space of the set of functions which are analytic in $\mathcal{D}$, where the topology is given by convergence that is uniform on compact subsets of $\mathcal{D}$. Convergence in the norm of $\mathcal{F}_\alpha$ implies convergence in H. We give the argument in the case $\alpha > 0$. First note that (1.1) implies

$$| f(z) | \leq \frac{\|\mu\|}{(1-r)^\alpha} \tag{1.33}$$

for $|z| \leq r$ ($0 < r < 1$). Suppose that the sequence $\{ f_n \}$ of functions in $\mathcal{F}_\alpha$ converges to f in the norm of $\mathcal{F}_\alpha$. Then there is a sequence $\{\mu_n\}$ and μ 0 $\mathcal{M}$ such that μ_n represents f_n, μ represents f, and $\| \mu_n - \mu \| \to 0$ as $n \to \infty$. The measure $\mu_n - \mu$ represents $f_n - f$ and hence (1.33) implies

$$\sup_{|z| \leq r} | f_n(z) - f(z) | \leq \frac{\|\mu_n - \mu\|}{(1-r)^\alpha} \tag{1.34}$$

for $|z| \le r$ $(0 < r < 1)$ and $n = 1, 2, \ldots$.
Hence

$$\sup_{|z| \le r} |f_n(z) - f(z)| \le \frac{\|f_n - f\|_{\mathcal{F}_\alpha}}{(1-r)^\alpha}.$$

Since $\|f_n - f\|_{\mathcal{F}_\alpha} \to 0$ this implies that $f_n \to f$ in the topology of H.

The following fact is another useful connection between the topologies on $\mathcal{F}_\alpha$ and on H.

Proposition 1.9 *For* $R \ge 0$ *let*

$$\mathcal{F}_\alpha(R) = \{f \in \mathcal{F}_\alpha : \|f\|_{\mathcal{F}_\alpha} \le R\}.$$

If $f_n \in \mathcal{F}_\alpha(R)$ *for* $n = 1, 2, \ldots$ *then there exist a subsequence* $\{f_{n_k}\}$ $(k = 1, 2, \ldots)$ *and* $f \in \mathcal{F}_\alpha(R)$ *such that* $f_{n_k} \to f$ *as* $k \to \infty$, *uniformly on each compact subset of* $\mathcal{D}$.

Proof: We give the argument in the case $\alpha > 0$. Since $\|f_n\|_{\mathcal{F}_\alpha} \le R$ for $n = 1, 2, \ldots$, Theorem 1.8 implies that there exists $\mu_n \in \mathcal{M}$ such that $\|\mu_n\| \le R$ and

$$f_n(z) = \int_T \frac{1}{(1-\bar{\zeta}z)^\alpha}\, d\mu_n(\zeta) \qquad (|z| < 1). \qquad (1.35)$$

By the Banach-Alaoglu theorem there exist a subsequence which we continue to call $\{\mu_n\}$ and $\mu \in \mathcal{M}$ such that $\mu_n \to \mu$ as $n \to \infty$. We have $\|\mu\| \le R$. Since $\mu_n \to \mu$,

$$\int_T \frac{1}{(1-\bar{\zeta}z)^\alpha}\, d\mu_n(\zeta) \to \int_T \frac{1}{(1-\bar{\zeta}z)^\alpha}\, d\mu(\zeta) \qquad (1.36)$$

for every z $(|z| < 1)$. Since $\|\mu_n\| \le R$, (1.33) implies that the sequence $\{f_n\}$ is locally bounded. Hence Montel's theorem implies there exist a subsequence of $\{f_n\}$, say $\{g_m\}$ $(m = 1, 2, \ldots)$, and f such that $g_m \to f$ uniformly on compact subsets of $\mathcal{D}$. Because of (1.36) this yields

$$f(z) = \int_{T} \frac{1}{(1-\overline{\zeta}z)^{\alpha}}\, d\mu(\zeta) \qquad (|z|<1). \tag{1.37}$$

Thus $f \in \mathcal{F}_\alpha$ and $\| f \|_{\mathcal{F}_\alpha} \leq R$ since $\| \mu \| \leq R$.

NOTES

The main developments in this book trace back to work initiated by Havin [1958, 1962] on the Cauchy transforms defined by (1.3). There is an extensive literature about Cauchy transforms where the measures are supported on various subsets of $\mathbb{C}$. The families $\mathcal{F}_\alpha$ for $\alpha > 0$ were introduced in MacGregor [1987]. This extends the families studied by Brickman, Hallenbeck, MacGregor and Wilken [1973] given by (1.1) where μ is a probability measure. Formula (1.1) occurs elsewhere, usually where μ is absolutely continuous with respect to Lebesgue measure. For example, this occurs for Dirichlet spaces as noted in Nagel, Rudin and J.H. Shapiro [1982]. The family $\mathcal{F}_0$ was introduced by Hibschweiler and MacGregor [1989]. For $\alpha < 0$, $\mathcal{F}_\alpha$ was defined by Hibschweiler and MacGregor [1993]. A survey about fractional Cauchy transforms can be found in MacGregor [1999].

When $\alpha = 1$ (or more generally when α is a positive integer) (1.1) defines a function which is analytic in $\mathbb{C} \setminus T$. In the case $\alpha = 1$, an analytic characterization of functions analytic in $\mathbb{C} \setminus T$ and having the form (1.1) is in Aleksandrov [1981; see section 5]. This result and its proof are given in Chapter 10. For every α, no such intrinsic analytic characterization is known for a function defined in $\mathbb{D}$ sufficient to imply that it has the representation (1.1).

Two references about H^p spaces and h^p spaces are Duren [1970] and Koosis [1980]. A proof of Theorem 1 as well as the uniqueness of the representing measure is in Duren [1983; see p. 22]. A proof of (1.10) is in Bourdon and Cima [1988]. Koosis [1980; see p. 40] contains a proof of Theorem 2. L. Brickman proved Theorem 8. The argument also is given by Cima and Siskakis [1999] for the case $\alpha = 1$. Most of the other results in this chapter are well established facts.

CHAPTER 2

Basic Properties of $\mathcal{F}_\alpha$

Preamble. We begin with a presentation of properties of the gamma function and the binomial coefficients. This forms a foundation for obtaining results about the families $\mathcal{F}_\alpha$. A product theorem is proved, namely, if $\alpha > 0$, $\beta > 0$, $f \in \mathcal{F}_\alpha$ and $g \in \mathcal{F}_\beta$ then $f \in \mathcal{F}_{\alpha+\beta}$. Also $f \in \mathcal{F}_\alpha$ if and only if $f' \in \mathcal{F}_{\alpha+1}$. The families $\mathcal{F}_\alpha$ are strictly increasing in α. Formulas are obtained which give mappings between $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$.

An analytic condition implying membership in $\mathcal{F}_\alpha$ for $\alpha > 1$ is given. This yields a condition sufficient to imply membership in $\mathcal{F}_\alpha$ when $\alpha > 0$. The latter condition provides a connection between the families $\mathcal{F}_\alpha$ and the Besov spaces $\mathcal{B}_\alpha$. Two applications are given. The first shows that if the moduli of the zeros of an infinite Blaschke product f are restricted in a precise way, then f belongs to $\mathcal{F}_\alpha$ for suitable α. As a second application, the singular inner function $S(z) = \exp\left[-\frac{1+z}{1-z}\right]$ belongs to $\mathcal{F}_\alpha$ for $\alpha > \frac{1}{2}$. Chapter 3 includes a more general study about the membership of inner functions in $\mathcal{F}_\alpha$.

The chapter concludes with a discussion of Hadamard products and their relationship to the family $\mathcal{F}_1$. It is shown that if $f \in \mathcal{F}_1$ and $g \in \mathcal{F}_1$, then $f * g \in \mathcal{F}_1$. Also, an analytic function f belongs to $\mathcal{F}_1$ if and only if $f * g \in H^\infty$ for every $g \in H^\infty$.

We begin with a number of lemmas, some of which involve the gamma function Γ and the binomial coefficients $A_n(\alpha)$. The first lemma lists some known facts.

Lemma 2.1 *For complex numbers* z *except the nonpositive integers,*

$$(z-1)\,\Gamma(z-1) = \Gamma(z) \tag{2.1}$$

and

$$\Gamma(z) = \lim_{k\to\infty} \frac{k^{z-1}\, k!}{z(z+1)\cdots(z+k-1)}. \tag{2.2}$$

If Re (z) > 0 *and* Re (w) > 0 *then*

$$\int_0^1 t^{z-1}\,(1-t)^{w-1}\,dt = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}. \tag{2.3}$$

The asymptotic expansion

$$(2\pi)^{-1/2}\, e^z\, z^{-z+1/2}\, \Gamma(z) \approx 1 + \sum_{k=1}^{\infty} \frac{a_k}{z^k} \tag{2.4}$$

as $|z| \to \infty$ *holds for* $|\arg z| \le \Phi$ *where* $0 < \Phi < \pi$.

Lemma 2.2 *If* n *is a positive integer,* α 0 | *and* $\alpha \neq 0, -1, -2, \ldots$, *then*

$$A_n(\alpha) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\, n!}. \tag{2.5}$$

Proof: Let n and α satisfy the stated conditions. Below we use (2.2) where $z = n + \alpha$, $z = \alpha$ and $z = n + 2$.

$$\begin{aligned}
&\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\,\Gamma(n+2)} \\
&= \lim_{k\to\infty}\left(\frac{1}{k\cdot k!}\,\frac{\alpha(\alpha+1)\cdots(\alpha+k-1)(n+2)(n+3)\cdots(n+k+1)}{(n+\alpha)(n+\alpha+1)\cdots(n+\alpha+k-1)}\right) \\
&= \frac{A_n(\alpha)}{n+1}\lim_{k\to\infty}\left(\frac{1}{k\cdot k!}\,\frac{(\alpha+n)(\alpha+n+1)\cdots(\alpha+k-1)}{(\alpha+n)(\alpha+n+1)\cdots(\alpha+n+k-1)}\,(n+k+1)!\right) \\
&= \frac{A_n(\alpha)}{n+1}\lim_{k\to\infty}\left(\frac{1}{k\cdot k!}\,\frac{(n+k+1)!}{(\alpha+k)(\alpha+k+1)\cdots(\alpha+k+n-1)}\right) \\
&= \frac{A_n(\alpha)}{n+1}\lim_{k\to\infty}\left(\frac{1}{k\cdot k!}\,\frac{(n+k+1)!}{k^n}\right) \\
&= \frac{A_n(\alpha)}{n+1}\lim_{k\to\infty}\left(\frac{(k+1)(k+2)\cdots(k+n+1)}{k^{n+1}}\right) = \frac{A_n(\alpha)}{n+1}.
\end{aligned}$$

This yields (2.5).

Lemma 2.3 *Suppose that* $u \in \mathbb{D}$, $v \in \mathbb{D}$, $\alpha > 0$ *and* $\beta > 0$. *Then*

$$\frac{1}{(1-u)^{\alpha}\,(1-v)^{\beta}} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^1 \frac{t^{\alpha-1}\,(1-t)^{\beta-1}}{[1-\{tu+(1-t)v\}]^{\alpha+\beta}}\,dt. \qquad (2.6)$$

Proof: We first show that

$$\frac{1}{(1-z)^{\gamma}} = \frac{\Gamma(\delta)}{\Gamma(\gamma)\,\Gamma(\delta-\gamma)} \int_0^1 \frac{t^{\gamma-1}\,(1-t)^{\delta-\gamma-1}}{(1-tz)^{\delta}}\,dt \qquad (2.7)$$

where $\delta > \gamma > 0$ and $z \notin [1, \infty)$. Since both sides of (2.7) define analytic functions in $\mathbb{C} \setminus [1, \infty)$ it suffices to prove (2.7) where $|z| < 1$. Suppose that $|z| < 1$ and $0 \le t \le 1$. Then

$$\frac{1}{(1-tz)^{\delta}} = \sum_{n=0}^{\infty} A_n(\delta)\, t^n z^n.$$

If we use this expansion on the right-hand side of (2.7) and integrate term by term we obtain (2.7) from (2.3) and Lemma 2.2.

Under the conditions of the lemma, the relation (2.7) holds where $z = (u-v)/(1-v)$, $\gamma = \alpha$, and $\delta = \alpha + \beta$. Therefore

$$\left[\frac{1-v}{1-u}\right]^{\alpha} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^1 \frac{t^{\alpha-1}\,(1-t)^{\beta-1}\,(1-v)^{\alpha+\beta}}{[1-\{tu+(1-t)v\}]^{\alpha+\beta}}\,dt. \qquad (2.8)$$

If both sides of (2.8) are divided by $(1-v)^{\alpha+\beta}$ this yields (2.6).

Lemma 2.4 *For* $\alpha > 0$ *there is an asymptotic expansion*

$$\frac{A_n(\alpha)}{n^{\alpha-1}} \approx \sum_{k=0}^{\infty} \frac{b_k}{n^k} \qquad (n \to \infty) \qquad (2.9)$$

and $b_0 = 1/\Gamma(\alpha)$.

Proof: We first apply (2.4) where $z = n + \alpha$ and where $z = n + 1$. Next we use the relation

$$\frac{1}{(n+\beta)^k} = \frac{1}{n^k}\left[1+\frac{\beta}{n}\right]^{-k}$$

for large values of n and the fact that $(1 + (\beta/n))^{-k}$ has a power series expansion in 1/n. This yields the asymptotic expansions

$$(2\pi)^{-\frac{1}{2}} e^{n+\alpha} (n+\alpha)^{-n-\alpha+\frac{1}{2}} \Gamma(n+\alpha) \approx 1 + \sum_{k=1}^{\infty} \frac{c_k}{n^k} \quad (n \to \infty) \qquad (2.10)$$

and

$$(2\pi)^{-\frac{1}{2}} e^{n+1} (n+1)^{-n-\frac{1}{2}} \Gamma(n+1) \approx 1 + \sum_{k=1}^{\infty} \frac{d_k}{n^k} \quad (n \to \infty) \qquad (2.11)$$

for suitable numbers c_k and d_k. Division of the left side of (2.10) by the left side of (2.11) yields an expansion

$$\frac{e^{\alpha-1}(n+1)^{n+\frac{1}{2}}\Gamma(n+\alpha)}{(n+\alpha)^{n+\alpha-\frac{1}{2}}\Gamma(n+1)} \approx 1 + \sum_{k=1}^{\infty} \frac{e_k}{n^k} \quad (n \to \infty) \qquad (2.12)$$

for suitable numbers e_k. If

$$\gamma_n = \frac{(n+1)^{n+\frac{1}{2}}}{(n+\alpha)^{n+\alpha-\frac{1}{2}}},$$

then rewriting γ_n as

$$\gamma_n = \frac{n^{\frac{1}{2}}\left[1+\frac{1}{n}\right]^{\frac{1}{2}}\left[1+\frac{1}{n}\right]^{n}}{n^{\alpha-\frac{1}{2}}\left[1+\frac{\alpha}{n}\right]^{\alpha-\frac{1}{2}}\left[1+\frac{\alpha}{n}\right]^{n}}$$

yields the expansion

$$n^{\alpha-1}\ e^{\alpha-1}\ \gamma_n \ \approx 1 + \sum_{k=1}^{\infty} \frac{f_k}{n^k} \qquad (n \to \infty) \tag{2.13}$$

for suitable numbers f_k. From (2.12) and (2.13) we obtain

$$n^{1-\alpha}\ \frac{\Gamma(n+\alpha)}{\Gamma(n+1)} \approx 1 + \sum_{k=1}^{\infty} \frac{g_k}{n^k} \qquad (n \to \infty) \tag{2.14}$$

for suitable numbers g_k. Hence (2.5) yields (2.9) and $b_0 = 1/\Gamma(\alpha)$.

Lemma 2.5 *Suppose that the function* F *is analytic in* $\overline{D}$ *and* $|w| \le 1$. *Let* $f(z) = F(wz)$ *for* $|z| < 1$. *Then there exists* μ 0 $\mathcal{M}^*$ *such that*

$$f(z) = \int_T F(\zeta z)\ d\mu(\zeta) \quad \textit{for } |z| < 1.$$

Proof: For a fixed z ($|z| < 1$) let $G(w) = F(zw)$ for $|w| \le 1$. Then G is analytic in $\overline{D}$ and the Poisson formula gives

$$G(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}\left(\frac{e^{i\theta} + w}{e^{i\theta} - w}\right) G(e^{i\theta})\ d\theta$$

for $|w| < 1$. In other words

$$F(wz) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}\left(\frac{e^{i\theta} + w}{e^{i\theta} - w}\right) F(e^{i\theta}z)\ d\theta$$

for $|w| < 1$ and $|z| < 1$. For $|w| < 1$,

$$\frac{1}{2\pi}\operatorname{Re}\left(\frac{e^{i\theta} + w}{e^{i\theta} - w}\right) d\theta$$

defines a probability measure on $[-\pi, \pi]$. This proves the result when $|w| < 1$. When $|w| = 1$ the conclusion follows by choosing μ to be unit point mass at w.

Lemma 2.6 *Let* $\alpha > 0$ *and* $\beta > 0$. *If* f 0 $\mathcal{F}_\alpha^*$ *and* g 0 $\mathcal{F}_\beta^*$, *then* $f \cdot g$ 0 $\mathcal{F}_{\alpha+\beta}^*$.

Proof: The hypotheses imply that the product $f \cdot g$ can be expressed as an integral with respect to a probability measure on $T \times T$ where the integrand has the form $(1-\overline{\zeta}z)^{-\alpha}(1-\overline{w}z)^{-\beta}$. The set $\mathcal{F}_{\alpha+\beta}^*$ is convex and closed with respect to the topology of H. Hence it suffices to show that each function

$$z \longmapsto \frac{1}{(1-\overline{\zeta}z)^{\alpha}(1-\overline{w}z)^{\beta}}$$

belongs to $\mathcal{F}_{\alpha+\beta}^*$ where $|\zeta| = |w| = 1$. Lemma 2.3 gives

$$\frac{1}{(1-\overline{\zeta}z)^{\alpha}(1-\overline{w}z)^{\beta}} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}}{[1-\sigma(t)z]^{\alpha+\beta}}\, dt \qquad (2.15)$$

where $\sigma(t) = t\overline{\zeta} + (1-t)\overline{w}$. Let ρ be the measure defined by

$$d\rho(t) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, t^{\alpha-1}(1-t)^{\beta-1}\, dt.$$

By (2.3), ρ is a probability measure on [0, 1]. Thus (2.15) implies that it suffices to show that each function

$$z \longmapsto \frac{1}{[1-\sigma(t)z]^{\alpha+\beta}}$$

belongs to $\mathcal{F}_{\alpha+\beta}^*$ for $|\zeta| = 1$, $|w| = 1$ and $0 \le t \le 1$. This follows from Lemma 2.5. □

Theorem 2.7 *Let* $\alpha > 0$ *and* $\beta > 0$. *If* f 0 $\mathcal{F}_\alpha$ *and* g 0 $\mathcal{F}_\beta$ *then* $f \cdot g$ 0 $\mathcal{F}_{\alpha+\beta}$ and

$$\| f \cdot g \|_{\mathcal{F}_{\alpha+\beta}} \le \| f \|_{\mathcal{F}_\alpha} \, \| g \|_{\mathcal{F}_\beta} .$$

Proof: The hypotheses imply that f is a linear combination of four functions in $\mathcal{F}_\alpha^*$ and g is a linear combination of four functions in $\mathcal{F}_\beta^*$. Hence $f \cdot g$ is a linear combination of sixteen functions of the form $h \cdot k$ where h 0 $\mathcal{F}_\alpha^*$ and

$k \in \mathrm{F}_\beta^*$. Lemma 2.6 implies that each product $h \cdot k$ belongs to $\mathrm{F}_{\alpha+\beta}^*$. Therefore $f \cdot g \in \mathcal{F}_{\alpha+\beta}$.

To prove the norm inequality let $f(z) = \displaystyle\int_T \frac{1}{(1-\overline{\zeta}z)^\alpha}\, d\mu(\zeta)$ and

$g(z) = \displaystyle\int_T \frac{1}{(1-\overline{\omega}z)^\beta}\, d\upsilon(\omega)$ for $|z| < 1$, where μ and υ belong to $\mathcal{M}$. If

$f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \displaystyle\sum_{n=0}^{\infty} b_n z^n$, then relation (1.18) yields

$$f(z)g(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} A_k(\alpha)\; A_{n-k}(\beta) \int_T \overline{\zeta}^k \; d\mu(\zeta) \int_T \overline{\omega}^{n-k}\; d\upsilon(\omega)\right) z^n$$

for $|z| < 1$.

Let $\mathcal{C}$ denote the Banach space of continuous complex-valued functions defined on T, where the norm is given by $\|h\|_{\mathcal{C}} = \sup\limits_{|x|=1} |h(x)|$ for $h \in \mathcal{C}$. For each $h \in \mathcal{C}$, extend h to $\overline{\mathrm{D}}$ by the Poisson integral in $\mathbb{D}$. Let ρ be the measure defined in the proof of Lemma 2.6 and let

$$H(\zeta, \omega) = \int_0^1 h(t\zeta + (1-t)\omega)\; d\rho(t) \qquad (|\zeta| = 1, |\omega| = 1).$$

Then H is continuous on $T \times T$ and

$$\sup_{|\zeta|=1,\, |\omega|=1} |H(\zeta,\omega)| = \|h\|_C.$$

Let $\varphi(h) = \displaystyle\int_{T\times T} H(\zeta, \omega)\; d(\mu\times\upsilon)\,(\zeta, \omega)$ for $h \in \mathcal{C}$. Then φ is linear and bounded since $|\varphi(h)| \le \|h\|_C\, \|\mu\|\, \|\upsilon\|$. By the Riesz representation theorem there exists $\lambda \in \mathcal{M}$ with

$$\varphi(h) = \int_T h(x)\; d\lambda(x)$$

and $\|\lambda\| \leq \|\mu\| \, \|\upsilon\|$. In particular if $h(x) = \overline{x}^n$ for x 0 T and n = 0, 1, 2, … , then

$$\int_T \overline{x}^n \, d\lambda(x) = \int_{T\times T} \int_0^1 (t\overline{\zeta} + (1-t)\overline{\omega})^n \, d\rho(t) \, d(\mu \times \upsilon)\,(\zeta, \omega).$$

A calculation involving the binomial theorem, (2.3) and (2.5) shows that

$$\int_0^1 (t\overline{\zeta} + (1-t)\,\overline{\omega})^n \, d\rho(t) = \frac{1}{A_n(\alpha+\beta)} \sum_{k=0}^{n} A_k(\alpha) \, A_{n-k}(\beta)\,\overline{\zeta}^k \, \overline{\omega}^{n-k}$$

for n = 0, 1, … . It follows that

$$\begin{aligned} f(z)\, g(z) &= \sum_{n=0}^{\infty} A_n(\alpha+\beta) \int_T \overline{x}^n \, d\lambda(x) \, z^n \\ &= \int_T \frac{1}{(1-\overline{x}z)^{\alpha+\beta}} \, d\lambda(x). \end{aligned}$$

Therefore $\| f \cdot g \|_{\mathcal{F}_{\alpha+\beta}} \leq \|\lambda\| \leq \|\mu\| \, \|\upsilon\|$. Since μ and υ are arbitrary measures representing f and g, the previous inequality yields $\| f \cdot g \|_{\mathcal{F}_{\alpha+\beta}} \leq \| f \|_{\mathcal{F}_\alpha} \, \| g \|_{\mathcal{F}_\beta}$. □

Theorem 2.8 *Let* $\alpha \geq 0$. *Then* f 0 $\mathcal{F}_\alpha$ *if and only if* f' 0 $\mathcal{F}_{\alpha+1}$. *Also there are positive constants* A *and* B *depending only on* α *such that*

$$\| f' \|_{\mathcal{F}_{\alpha+1}} \leq A \, \| f \|_{\mathcal{F}_\alpha}$$

and

$$\| f \|_{\mathcal{F}_\alpha} \leq | f(0) | + B \, \| f' \|_{\mathcal{F}_{\alpha+1}} .$$

Proof: We first assume f 0 $\mathcal{F}_\alpha$ where $\alpha > 0$. Then (1.1) implies

$$f'(z) = \int_T \frac{1}{(1-\overline{\zeta}z)^{\alpha+1}} \, d\upsilon(\zeta)$$

where $d\upsilon(\zeta) = \alpha\bar{\zeta}\, d\mu(\zeta)$. Therefore $f' \ 0 \ \mathcal{F}_{\alpha+1}$ and since $\|\upsilon\| = \alpha\,\|\mu\|$ we have $\|f'\|_{\mathcal{F}_{\alpha+1}} \le \alpha\,\|f\|_{\mathcal{F}_\alpha}$.

Conversely, suppose that $f' \ 0 \ \mathcal{F}_\beta$ and $\beta > 1$. Then

$$f'(z) = \int_T \frac{1}{(1 - \bar{\zeta}z)^\beta}\, d\mu(\zeta) \qquad (|z| < 1)$$

for some $\mu \ 0 \ \mathcal{M}$. Since $f(z) = f(0) + \int_0^z f'(w)\, dw$, this implies that

$$f(z) = f(0) + b + \int_T \frac{1}{(1 - \bar{\zeta}z)^{\beta-1}}\, d\upsilon(\zeta)$$

where $b = \left[1/(1-\beta)\right] \int_T \zeta\, d\mu(\zeta)$ and $d\upsilon(\zeta) = \left[\zeta/(\beta-1)\right] d\mu(\zeta)$. Hence

$$f(z) = \int_T \frac{1}{(1 - \bar{\zeta}z)^{\beta-1}}\, d\sigma(\zeta)$$

where $\sigma = (f(0) + b)\,\lambda + \upsilon$ and λ is the measure $\frac{d\theta}{2\pi}$ $(\zeta = e^{i\theta})$. Therefore $f \ 0 \ \mathcal{F}_{\beta-1}$. Also

$$\|\sigma\| \le |f(0)| + \frac{2}{\beta - 1}\|f'\|_{\mathcal{F}_\beta},$$

which yields

$$\|f\|_{\mathcal{F}_{\beta-1}} \le |f(0)| + \frac{2}{\beta - 1}\|f'\|_{\mathcal{F}_\beta}.$$

This completes the proof in the case $\alpha > 0$. A similar argument applies when $\alpha = 0$.

Lemma 2.9 *If* $0 < \alpha < \beta$ *then* $\mathcal{F}_\alpha^* \subset \mathcal{F}_\beta^*$.

Proof: Let $\gamma > 0$, and let $I(z) = 1$ $(|z| < 1)$. Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{(1 - e^{-i\theta}z)^{\gamma}}\, d\theta = 1$$

for $|z| < 1$ and since $1/(2\pi)\, d\theta$ defines a probability measure, $I \; 0 \; F_{\gamma}^{*}$ for every $\gamma > 0$.

Suppose that $0 < \alpha < \beta$ and $f \; 0 \; F_{\alpha}^{*}$. Let $\gamma = \beta - \alpha$. Then $I \; 0 \; F_{\gamma}^{*}$ and hence Lemma 2.6 yields $f = f \cdot I \; 0 \; F_{\alpha+\gamma}^{*}$. Therefore $F_{\alpha}^{*} \subset F_{\beta}^{*}$.

Theorem 2.10 *If* $0 \leq \alpha < \beta$ *then* $\mathcal{F}_{\alpha} \; \delta \; \mathcal{F}_{\beta}$ *and* $\mathcal{F}_{\alpha} \neq \mathcal{F}_{\beta}$.

Proof: Let $0 < \alpha < \beta$ and assume that $f \; 0 \; \mathcal{F}_{\alpha}$. Let I be defined as in the proof of Lemma 2.9. Then $I \; 0 \; F_{\gamma}^{*}$ for every $\gamma > 0$ and hence $\| I \|_{F_{\beta-\alpha}} = 1$. Theorem 2.7 implies that $f = f \cdot I \; 0 \; \mathcal{F}_{\beta}$ and $\| f \|_{F_{\beta}} \leq \| f \|_{F_{\alpha}}$.

Next consider the case where $\alpha = 0$ and $\beta > 0$. Assume that $f \; 0 \; \mathcal{F}_{0}$. Theorem 2.8 implies that $f' \; 0 \; \mathcal{F}_{1}$. The previous case of this theorem yields $f' \; 0 \; \mathcal{F}_{\beta+1}$. Hence Theorem 2.8 implies $f \; 0 \; \mathcal{F}_{\beta}$.

It remains to show that $\mathcal{F}_{\alpha} \neq \mathcal{F}_{\beta}$. The relations (1.1) and (1.2) imply that if $f \; 0 \; \mathcal{F}_{\alpha}$ $(\alpha > 0)$ and $g \; 0 \; \mathcal{F}_{0}$, then

$$|f(z)| = O\left(\frac{1}{(1 - |z|)^{\alpha}}\right)$$

and

$$|g(z) - g(0)| = O\left(\log \frac{1}{1 - |z|}\right).$$

The function $z \mapsto (1 - z)^{-\beta}$ belongs to $\mathcal{F}_{\beta}$ but does not satisfy these growth conditions when $0 \leq \alpha < \beta$.

Theorem 2.11 *Let* $F_{\infty} = \bigcup_{\alpha>0} F_{\alpha}$. *Then* $f \; 0 \; \mathcal{F}_{\infty}$ *if and only if f is analytic in* D *and there exist positive constants* A *and* α *such that*

$$|f(z)| \leq \frac{A}{(1 - |z|)^{\alpha}} \quad \textit{for } |z| < 1. \qquad (2.16)$$

Proof: Suppose that $f \in \mathcal{F}_\infty$. Then $f \in \mathcal{F}_\alpha$ for some $\alpha > 0$. Hence (2.16) follows from (1.1), where $A = \| \mu \|$.

Conversely, suppose that f is analytic in $\mathcal{D}$ and (2.16) holds. We first assume $0 < \alpha < 1$. Let

$$f_1(z) = \int_0^z f(w)\, dw \qquad (|z| < 1).$$

Then (2.16) implies that if $|z| = r$ $(0 < r < 1)$ then

$$| f_1(z)| \leq \int_0^1 \frac{A}{(1 - tr)^\alpha}\, dt \leq \frac{A}{1 - \alpha}.$$

Hence $f_1 \in H^\infty$ and then $f_1 \in \mathcal{F}_1$. Because $f = f_1'$ Theorem 2.8 implies $f \in \mathcal{F}_2$. Therefore $f \in \mathcal{F}_\infty$.

Next we assume $1 \leq \alpha < 2$. Note that if (2.16) holds for $\alpha = 1$, then it holds for any $\alpha > 1$. Thus we may assume $1 < \alpha < 2$. Now the function f_1 defined above satisfies

$$| f_1(z)| \leq \frac{A}{\alpha - 1} \frac{1}{(1 - r)^{\alpha - 1}}$$

for $|z| \leq r$ $(0 < r < 1)$. Let $f_2(z) = \int_0^z f_1(w)\, dw$ for $|z| < 1$. Because $\alpha < 2$ the estimate on f_1 implies that $f_2 \in H^\infty$. Thus $f_2 \in \mathcal{F}_1$ and Theorem 2.8 yields $f_1 = f_2' \in \mathcal{F}_2$. Hence Theorem 2.8 shows that $f = f_1' \in \mathcal{F}_3$. Therefore $f \in \mathcal{F}_\infty$.

In general we argue as follows. Suppose that (2.16) holds and $\alpha > 0$. Let $n = [\alpha] + 1$. Let $f_0 = f$ and for $k = 1, 2, \ldots, n$ let $f_k(z) = \int_0^z f_{k-1}(w)\, dw$. By successively obtaining estimates on $f_0, f_1, \ldots, f_n$ we find that $f_n \in H^\infty$. Hence $f_n \in \mathcal{F}_1$ and applications of Theorem 2.8 yield $f_{n-1} \in \mathcal{F}_2$, $f_{n-2} \in \mathcal{F}_3, \ldots, f_1 \in \mathcal{F}_n$ and $f_0 = f \in \mathcal{F}_{n+1}$. Therefore $f \in \mathcal{F}_\infty$.

Let $\alpha > 0$ and $\beta > 0$. The mapping $L_{\alpha,\beta} : H \to H$ is defined by $f \to g$ where

$f(z) = \sum_{n=0}^{\infty} a_n z^n \ (|z| < 1)$ and

$$g(z) = \sum_{n=0}^{\infty} \frac{A_n(\beta)}{A_n(\alpha)} a_n z^n \qquad (|z| < 1). \tag{2.17}$$

Since $A_n(\alpha) \neq 0$ and $\lim\limits_{n\to\infty} A_n(\gamma)^{1/n} = 1$ for every $\gamma > 0$ by (2.9), we see that f 0 H implies that g 0 H. Also $L_{\alpha,\beta}$ is a linear homeomorphism on H. If $\alpha > \beta$ then (2.17) can be expressed as

$$g(z) = \frac{\Gamma(\alpha)}{\Gamma(\beta)\,\Gamma(\alpha-\beta)} \int_0^1 t^{\beta-1}\,(1-t)^{\alpha-\beta-1}\, f(tz)\,dt \tag{2.18}$$

for $|z| < 1$. To show this we use (2.3) and (2.5) as follows.

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{\Gamma(\beta)\,\Gamma(\alpha-\beta)} \int_0^1 t^{\beta-1}\,(1-t)^{\alpha-\beta-1}\, f(tz)\,dt \\
&= \frac{\Gamma(\alpha)}{\Gamma(\beta)\,\Gamma(\alpha-\beta)} \int_0^1 \left\{ t^{\beta-1}\,(1-t)^{\alpha-\beta-1} \sum_{n=0}^{\infty} a_n t^n z^n \right\} dt \\
&= \sum_{n=0}^{\infty} \left(\frac{\Gamma(\alpha)}{\Gamma(\beta)\,\Gamma(\alpha-\beta)} \int_0^1 t^{n+\beta-1}\,(1-t)^{\alpha-\beta-1}\,dt \right) a_n z^n \\
&= \sum_{n=0}^{\infty} \left(\frac{\Gamma(\alpha)}{\Gamma(\beta)\,\Gamma(\alpha-\beta)} \frac{\Gamma(n+\beta)\,\Gamma(\alpha-\beta)}{\Gamma(n+\alpha)} \right) a_n z^n \\
&= \sum_{n=0}^{\infty} \left(\frac{\Gamma(\alpha)}{\Gamma(\beta)} \frac{\Gamma(n+\beta)}{\Gamma(n+\alpha)} \right) a_n z^n \\
&= \sum_{n=0}^{\infty} \left(\frac{\Gamma(\alpha)\, n!}{\Gamma(n+\alpha)} \frac{\Gamma(n+\beta)}{\Gamma(\beta)\, n!} \right) a_n z^n \\
&= g(z).
\end{aligned}$$

From (1.18) and (2.17) we see that $L_{\alpha,\beta}$ gives a one-to-one mapping of $\mathcal{F}_\alpha$ onto $\mathcal{F}_\beta$, with f and g represented by the same collection of measures.

The relation between f and g defined by (2.17) can be expressed another way when $\beta = 1$. Suppose that $\alpha > 0$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(|z| < 1)$ and

$g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (|z| < 1)$ where $b_n = a_n / A_n(\alpha)$ for $n = 0, 1, 2, \ldots$. Assume that $|z| < r$ and $0 < r < 1$. Then

$$
\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} A_n(\alpha)\, b_n z^n \\
&= \sum_{n=0}^{\infty} A_n(\alpha) \left(\frac{1}{2\pi i} \int_{|\zeta| = r} \frac{g(\zeta)}{\zeta^{n+1}}\, d\zeta \right) z^n \\
&= \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{g(\zeta)}{\zeta} \left(\sum_{n=0}^{\infty} A_n(\alpha) \left(\frac{z}{\zeta}\right)^n \right) d\zeta \\
&= \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{g(\zeta)}{\zeta} \frac{1}{(1 - z/\zeta)^\alpha}\, d\zeta.
\end{aligned}
$$

By Cauchy's theorem this gives

$$
f(z) = \frac{1}{2\pi i} \int_{\Psi} \frac{g(\zeta)}{\zeta\, (1 - z/\zeta)^\alpha}\, d\zeta \qquad (2.19)
$$

where Ψ is any simple closed curve in $\mathcal{D}$ containing z in its interior. In particular, (2.19) gives a formula for f 0 $\mathcal{F}_\alpha$ in terms of g 0 $\mathcal{F}_1$.

The next theorem provides a sufficient condition for membership in $\mathcal{F}_\alpha$ when $\alpha > 1$. The proof of this theorem depends on the special case of (2.18) where $\beta = 1$ and $\alpha > 1$, namely,

$$
g(z) = (\alpha - 1) \int_0^1 (1 - t)^{\alpha - 2}\, f(tz)\, dt. \qquad (2.20)
$$

Theorem 2.12 *Let* $\alpha > 1$. *Suppose that the function f is analytic in* $\mathcal{D}$ *and*

$$
A \equiv \int_0^1 \int_{-\pi}^{\pi} |f(re^{i\theta})|\, (1 - r)^{\alpha - 2}\, d\theta\, dr < \infty.
$$

Then $f \in \mathcal{F}_\alpha$ *and* $\| f \|_{\mathcal{F}_\alpha} \le \dfrac{\alpha - 1}{2\pi} A.$

Proof: Let g be defined by (2.20) for $|z| < 1$. If $0 < r < 1$, then

$$\begin{aligned}\int_{-\pi}^{\pi} |g(re^{i\theta})|\, d\theta &\le (\alpha - 1) \int_0^1 \left(\int_{-\pi}^{\pi} |f(tre^{i\theta})|\, d\theta \right) (1-t)^{\alpha-2}\, dt \\ &\le (\alpha - 1) \int_0^1 \left(\int_{-\pi}^{\pi} |f(te^{i\theta})|\, d\theta \right) (1-t)^{\alpha-2}\, dt \\ &= (\alpha - 1) A < \infty.\end{aligned}$$

Therefore $g \in H^1$. It follows that $g \in \mathcal{F}_1$ and $\| g \|_{\mathcal{F}_1} \le \| g \|_{H^1} \le \dfrac{\alpha - 1}{2\pi} A.$ Because $g \in \mathcal{F}_1$ the earlier observations about the relation (2.20) imply that $f \in \mathcal{F}_\alpha$ and the measures representing g in $\mathcal{F}_1$ are the same as the measures representing f in $\mathcal{F}_\alpha$. Therefore $\| f \|_{\mathcal{F}_\alpha} \le (\alpha - 1)A/2\pi$.

Next we relate $\mathcal{F}_\alpha$ to a certain Besov space. For $\alpha > 0$, let $\mathcal{B}_\alpha$ denote the set of functions f that are analytic in D and satisfy

$$\int_0^1 \int_{-\pi}^{\pi} |f'(re^{i\theta})|\, (1 - r)^{\alpha-1}\, d\theta\, dr < \infty. \tag{2.21}$$

For $f \in \mathcal{B}_\alpha$, let

$$\| f \|_{\mathcal{B}_\alpha} \equiv |f(0)| + \int_0^1 \int_{-\pi}^{\pi} |f'(re^{i\theta})|\, (1 - r)^{\alpha-1}\, d\theta\, dr. \tag{2.22}$$

It can be shown that (2.22) defines a norm on $\mathcal{B}_\alpha$ and that $\mathcal{B}_\alpha$ is a Banach space with respect to this norm. Also if $0 < \alpha < \beta$ then $\mathcal{B}_\alpha \subset \mathcal{B}_\beta$.

The proof of the next lemma uses the following classical result of E. Cesàro (see Pólya and Szegö, vol. I [1972], p. 16): Suppose that $\{a_n\}$ and $\{b_n\}$ $(n = 0, 1, \ldots)$ are sequences such that $b_n > 0$ for $n = 0, 1, \ldots$, $\sum_{n=0}^{\infty} b_n z^n$ converges for $|z| < 1$ and $\sum_{n=0}^{\infty} b_n z^n$ diverges for $z = 1$. If $\lim_{n\to\infty} \dfrac{a_n}{b_n}$ exists and equals c, then $\sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < 1$ and

$$\lim_{r \to 1-} \left\{ \frac{\sum_{n=0}^{\infty} a_n r^n}{\sum_{n=0}^{\infty} b_n r^n} \right\}$$

exists and equals c.

Lemma 2.13 *For* $0 \le r < 1$ *and* $\gamma > 0$ *let*

$$I_\gamma(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|1 - re^{i\theta}|^\gamma} \, d\theta.$$

There are positive constants C_γ *depending only on* γ *such that the following limits hold.*

$$\textit{If } 0 < \gamma < 1, \textit{ then } \lim_{r \to 1-} I_\gamma(r) = C_\gamma. \tag{2.23}$$

$$\lim_{r \to 1-} \frac{I_1(r)}{\log \frac{1}{1-r}} = C_1. \tag{2.24}$$

$$\textit{If } \gamma > 1, \textit{ then } \lim_{r \to 1-} (1 - r)^{\gamma - 1} I_\gamma(r) = C_\gamma. \tag{2.25}$$

Proof: Note that

$$I_\gamma(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{(1 - re^{i\theta})^{\gamma/2}} \right|^2 d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{n=0}^{\infty} A_n(\gamma/2) r^n \; e^{in\theta} \right|^2 d\theta$$

$$= \sum_{n=0}^{\infty} \left(A_n(\gamma/2)\right)^2 \; r^{2n}.$$

First suppose that $\gamma > 1$ and let $J_\gamma(r) = \dfrac{1}{(1-r^2)^{\gamma-1}}$, $0 \le r < 1$. Then $J_\gamma(r) = \sum_{n=0}^{\infty} A_n(\gamma-1) \; r^{2n}$. Equation (2.9) yields

$$\lim_{n\to\infty} n^{2-\gamma} \; A_n(\gamma-1) = \frac{1}{\Gamma(\gamma-1)}.$$

Since $\gamma > 1$ this implies that the series for $J_\gamma(r)$ diverges when $r = 1$. Therefore Cesàro's result and (2.9) give

$$\lim_{r\to 1-} \frac{I_\gamma(r)}{J_\gamma(r)} = \frac{\Gamma(\gamma-1)}{\Gamma^2(\gamma/2)}.$$

This proves (2.25) where $C_\gamma = \dfrac{\Gamma(\gamma-1)}{2^{\gamma-1}\Gamma^2(\gamma/2)}$.

Next let $J_1(r) = \log \dfrac{1}{1-r^2} = \sum_{n=1}^{\infty} \dfrac{1}{n} r^{2n}$ for $0 \le r < 1$. Cesàro's result yields a comparison between $I_1(r)$ and $J_1(r)$ and the relation (2.24) follows, with $C_1 = 1/\pi$.

Finally if $0 < \gamma < 1$, then equation (2.9) implies that

$$\lim_{n\to\infty} n^{2-\gamma} \; A_n^2\left(\frac{\gamma}{2}\right) = \frac{1}{\Gamma^2(\gamma/2)}$$

and hence $\sum_{n=0}^{\infty} A_n^2(\gamma/2)$ converges when $0 < \gamma < 1$. Therefore if $0 < \gamma < 1$ then

$\lim_{r\to 1-} I_r(\gamma)$ exists and equals C_γ, where $C_\gamma = \sum_{n=0}^{\infty} A_n^2(\gamma/2)$. This proves (2.23). □

Lemma 2.13 implies that the following inequalities hold for $0 \le r < 1$ where A_γ is a positive constant depending only on γ.

$$I_\gamma(r) \le \begin{cases} A_\gamma & \text{if } 0 < \gamma < 1 \\ A_1 \log \dfrac{2}{1-r} & \text{if } \gamma = 1 \\ A_\gamma (1-r)^{1-\gamma} & \text{if } \gamma > 1. \end{cases} \tag{2.26}$$

Theorem 2.14 *If f 0 $\mathcal{B}_\alpha$ then f 0 $\mathcal{F}_\alpha$ and $\| f \|_{\mathcal{F}_\alpha} \le C \| f \|_{\mathcal{B}_\alpha}$ where C is a positive constant depending only on α. If f 0 $\mathcal{F}_\alpha$ then f 0 $\mathcal{B}_\beta$ for every β > α.*

Proof: Let f 0 $\mathcal{B}_\alpha$. If we apply Theorem 2.12 with f replaced by f' and α replaced by α + 1, then (2.21) implies that f' 0 $\mathcal{F}_{\alpha+1}$ and $\| f' \|_{\mathcal{F}_{\alpha+1}} \le \dfrac{\alpha}{2\pi} \| f \|_{\mathcal{B}_\alpha}$. Hence Theorem 2.8 shows that f 0 $\mathcal{F}_\alpha$ and

$$\| f \|_{\mathcal{F}_\alpha} \le | f(0)| + \frac{B\,\alpha}{2\pi} \| f \|_{\mathcal{B}_\alpha} \le C \| f \|_{\mathcal{B}_\alpha}$$

where C depends only on α.

To prove the second assertion, suppose that f 0 $\mathcal{F}_\alpha$ and α > 0. Then

$$f'(z) = \int_T \frac{\alpha\overline{\zeta}}{(1-\overline{\zeta}z)^{\alpha+1}}\, d\mu(\zeta)$$

for some μ 0 $\mathcal{M}$. If $0 \le r < 1$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} | f'(re^{i\theta}) |\, d\theta \le \alpha \int_T \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|1-\overline{\zeta}re^{i\theta}|^{\alpha+1}}\, d\theta\, d|\mu|(\zeta)$$

$$= \alpha \int_T \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|1-re^{i\theta}|^{\alpha+1}}\, d\theta\, d|\mu|(\zeta).$$

Hence (2.26) implies that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(re^{i\theta})|\, d\theta \le \frac{A}{(1-r)^{\alpha}} \|\mu\| \tag{2.27}$$

where A is a positive constant. Suppose that $\beta > \alpha$. Then (2.27) yields

$$\int_0^1 \int_{-\pi}^{\pi} |f'(re^{i\theta})|\,(1-r)^{\beta-1}\, d\theta\, dr \le 2\pi A\, \|\mu\| \int_0^1 \frac{1}{(1-r)^{\alpha-\beta+1}}\, dr < \infty.$$

Therefore $f \in \mathcal{B}_\beta$. □

To see that $\mathcal{B}_\alpha$ is a proper subset of $\mathcal{F}_\alpha$, let $f(z) = 1/(1-z)^{\alpha}$ and use the relation (2.25) where $\gamma = \alpha + 1$. If $f \in \mathcal{B}_\alpha$ then f can be represented in $\mathcal{F}_\alpha$ by a measure which is absolutely continuous with respect to Lebesgue measure. This follows from the fact that the function g in the argument for Theorem 2.12 belongs to H^1.

Each function in H^∞ belongs to $\mathcal{F}_1$ but not necessarily to $\mathcal{F}_\alpha$ for some $\alpha < 1$. To see this, let

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n^2} z^{2^n}$$

for $|z| < 1$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, $f \in H^\infty$. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$. In general, if $\alpha > 0$ and $f \in \mathcal{F}_\alpha$ then the relations (1.18) and (2.9) imply

$$|a_k| \le A\, k^{\alpha-1} \tag{2.28}$$

for $k = 1, 2, \ldots$ where A is a positive constant. In our example we have

$$a_k = \left(\frac{\log 2}{\log k}\right)^2$$

where $k = 2^n$ and $n = 1, 2, \ldots$. Thus (2.28) is not satisfied when $0 < \alpha < 1$.

An infinite Blaschke product is a function f which has the form

$$f(z) = c\, z^m \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z}_n z} \tag{2.29}$$

for $|z| < 1$ where $|c| = 1$, m is a nonnegative integer, $0 < |z_n| < 1$ for n = 1, 2, …, and

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty. \tag{2.30}$$

The infinite product converges because of (2.30), and $|f(z)| < 1$ for $|z| < 1$. Hence f 0 $\mathcal{F}_1$. If the zeros of f have a certain further restriction then f 0 $\mathcal{F}_\alpha$ for suitable α with $0 < \alpha < 1$. The argument uses the following lemma.

Lemma 2.15 *Let* $0 < \alpha < 1$ *and for* $0 \le x < 1$ *let*

$$F(x) = \int_0^1 \frac{1}{(1-r)^{1-\alpha}\,(1-rx)}\,dr. \tag{2.31}$$

There is a positive constant A *depending only on* α *such that*

$$F(x) \le \frac{A}{(1-x)^{1-\alpha}} \tag{2.32}$$

for $0 \le x < 1$.

Proof: From (2.3) and (2.1) we obtain

$$\int_0^1 (1-r)^{\alpha-1}\, r^n\, dr = \frac{n!}{\alpha(\alpha+1)\cdots(\alpha+n)} \tag{2.33}$$

for n = 0, 1, … and $\alpha > 0$. Hence, if $|x| < 1$ and $\alpha > 0$ then

$$\begin{aligned} F(x) &= \int_0^1 (1-r)^{\alpha-1} \sum_{n=0}^{\infty} r^n x^n\, dr \\ &= \sum_{n=0}^{\infty} \left(\int_0^1 (1-r)^{\alpha-1}\, r^n\, dr \right) x^n = \sum_{n=0}^{\infty} \frac{n!}{\alpha(\alpha+1)\cdots(\alpha+n)}\, x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{A_n(\alpha)\,(\alpha+n)}\, x^n. \end{aligned}$$

Since $0 < \alpha < 1$, Lemma 2.4 implies that

$$\lim_{n\to\infty} \frac{1}{A_n(\alpha)\,(\alpha + n)\,A_n(1-\alpha)} = \Gamma(\alpha)\,\Gamma(1-\alpha).$$

Also

$$\frac{1}{(1-x)^{1-\alpha}} = \sum_{n=0}^{\infty} A_n(1-\alpha)\,x^n.$$

Letting $a_n = \dfrac{1}{A_n(\alpha)(\alpha+n)}$ and $b_n = A_n(1-\alpha)$ we have $b_n > 0$, $\sum\limits_{n=1}^{\infty} b_n = \infty$ and

$$\frac{a_n}{b_n} \to \Gamma(\alpha)\,\Gamma(1-\alpha).$$

By the result of Cesàro stated before Lemma 2.13, it follows that

$$\lim_{x\to 1-} \left[\frac{F(x)}{\dfrac{1}{(1-x)^{1-\alpha}}} \right] = \Gamma(\alpha)\,\Gamma(1-\alpha).$$

This implies (2.32).

Theorem 2.16 *Let* $0 < \alpha < 1$. *Suppose that* $0 < |z_n| < 1$ *for* $n = 1, 2, \ldots$ *and*

$$\sum_{n=1}^{\infty} (1 - |z_n|)^{\alpha} < \infty. \tag{2.34}$$

Let $|c| = 1$ *and let* m *be a nonnegative integer. If f is the Blaschke product defined by* (2.29), *then* $f \in \mathcal{F}_\alpha$.

Proof: Logarithmic differentiation of (2.29) gives

$$\frac{f'(z)}{f(z)} = \frac{m}{z} + \sum_{n=1}^{\infty} \frac{|z_n|^2 - 1}{(z_n - z)(1 - \overline{z}_n z)} \tag{2.35}$$

for $0 < |z| < 1$ and $z \notin \{z_n\}$. Let

$$f_0(z) = \frac{m}{z} \, f(z)$$

and for n = 1, 2, … let

$$f_n(z) = c \, z^m \prod_{k \neq n} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z}_k z}.$$

Then (2.35) implies

$$f'(z) = f_0(z) + \sum_{n=1}^{\infty} f_n(z) \frac{|z_n|}{z_n} \frac{|z_n|^2 - 1}{(1 - \overline{z}_n z)^2} \tag{2.36}$$

for $|z| < 1$. By Schwarz's Lemma $| f_0(z) | \leq m$. Hence $| f_n(z) | \leq 1$ yields

$$| f'(z)| \leq m + \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|1 - \overline{z}_n z|^2}. \tag{2.37}$$

Let $0 \leq r < 1$. Then (2.37) implies

$$I \equiv \int_0^1 \int_{-\pi}^{\pi} | f'(re^{i\theta}) | (1 - r)^{\alpha - 1} \, d\theta \, dr$$

$$\leq \int_0^1 \left\{ 2\pi m + \sum_{n=1}^{\infty} \left(1 - |z_n|^2\right) \int_{-\pi}^{\pi} \frac{1}{|1 - \overline{z}_n re^{i\theta}|^2} \, d\theta \right\} (1 - r)^{\alpha - 1} \, dr$$

$$= 2\pi m \int_0^1 (1 - r)^{\alpha - 1} \, dr + \int_0^1 \sum_{n=1}^{\infty} (1 - |z_n|^2) \frac{2\pi}{1 - |z_n|^2 \, r^2} (1 - r)^{\alpha - 1} \, dr$$

$$\leq \frac{2\pi m}{\alpha} + 4\pi \sum_{n=1}^{\infty} (1 - |z_n|) \int_0^1 \frac{1}{(1 - r)^{1 - \alpha} (1 - |z_n| \, r)} \, dr.$$

Lemma 2.15 yields

$$I \leq \frac{2\pi m}{\alpha} + 4\pi \, A \sum_{n=1}^{\infty} (1 - |z_n|)^{\alpha}.$$

Hence our assumption (2.34) implies that $I < \infty$ and therefore f 0 $\mathcal{B}_\alpha$. Theorem 2.14 completes the argument.

We give another application of Theorem 2.14. It concerns the function

$$S(z) = \exp\left[-\frac{1+z}{1-z}\right]$$

which plays an important role in the study of H^p spaces. Since $|S(z)| < 1$ for $|z| < 1$ it follows that S 0 $\mathcal{F}_1$. We improve this by showing that S 0 $\mathcal{F}_\alpha$ for all $\alpha > ½$. Our argument uses the following elementary lemma.

Lemma 2.17 *Let* $z = re^{i\theta}$ *where* $0 \le r < 1$ *and* $-\pi \le \theta \le \pi$. *There are positive constants* A, B *and* C *such that the following statements are valid.*

(a) $|1 - z| \ge A\,|\theta|$.
(b) *If* $|\theta| \le 1 - r$ *then* $|1 - z| \le B\,(1 - r)$.
(c) *If* $1 - r \le |\theta| \le \pi$ *then* $|1 - z| \le C\,|\theta|$.

Proof: To prove these inequalities we may assume that $0 \le \theta \le \pi$. For such θ, let $y = |1 - z|^2 = 1 - 2r\cos\theta + r^2$. If $\theta = 0$, then $y = (1 - r)^2 > 0$. If $0 < \theta < \pi/2$, then y has a minimum at $r = \cos\theta$ and hence $y \ge \sin^2\theta$. If $\pi/2 \le \theta \le \pi$, then y has a minimum at $r = 0$ and hence $y \ge 1$. Since $\sin\theta \ge (2/\pi)\,\theta$ for $0 \le \theta \le \pi/2$, the cases above imply that $y \ge \theta^2/\pi^2$ for $0 \le \theta \le \pi$. This holds for all r, $0 \le r < 1$, and hence (a) follows.

Assume that $|\theta| \le 1 - r$. Since $1 - 2r\cos\theta + r^2 = (1 - r)^2 + 4r\sin^2(\theta/2)$ and since $\sin\varphi \le \varphi$ for $0 \le \varphi \le \pi/2$, this yields

$$|1 - z|^2 = (1 - r)^2 + 4r\sin^2(\theta/2) \le 2\,(1 - r)^2.$$

This proves (b).

Finally, assume $1 - r \le \theta$. Then

$$|1 - z|^2 = (1 - r)^2 + 4r\sin^2(\theta/2) \le 2\,\theta^2.$$

This implies (c). □

Theorem 2.18 *Let the function* S *be defined by*

$$S(z) = \exp\left[-\frac{1+z}{1-z}\right]$$

for $|z| < 1$. *Then* $S \in \mathcal{F}_\alpha$ *for* $\alpha > ½$ *and* $S \notin \mathcal{F}_\alpha$ *for* $\alpha < ½$.

Proof: Let $z = re^{i\theta}$ where $0 \le r < 1$ and $-\pi \le \theta \le \pi$. Since

$$S'(z) = \frac{-2}{(1-z)^2} S(z)$$

it follows that

$$\int_{-\pi}^{\pi} |S'(re^{i\theta})|\, d\theta = 2\int_{0}^{\pi} |S'(re^{i\theta})\,|\, d\theta = 4\,\{I(r) + J(r)\},$$

where

$$I(r) = \int_{0}^{1-r} \frac{1}{|1-re^{i\theta}|^2} \exp\left\{-\frac{1-r^2}{|1-re^{i\theta}|^2}\right\} d\theta \tag{2.38}$$

and

$$J(r) = \int_{1-r}^{\pi} \frac{1}{|1-re^{i\theta}|^2} \exp\left\{-\frac{1-r^2}{|1-re^{i\theta}|^2}\right\} d\theta. \tag{2.39}$$

From (b) in Lemma (2.17) we obtain

$$I(r) \le \int_{0}^{1-r} \frac{1}{(1-r)^2} \exp\left\{-\frac{1-r^2}{B^2(1-r)^2}\right\} d\theta$$

and hence

$$I(r) \le \frac{1}{1-r} \exp\left\{-\frac{1}{B^2(1-r)}\right\}. \tag{2.40}$$

From (a) and (c) in Lemma (2.17) we obtain

$$J(r) \le \int_{1-r}^{\pi} \frac{1}{A^2\theta^2} \exp\left\{-\frac{1-r}{C^2\theta^2}\right\} d\theta.$$

The substitution $x = \sqrt{1-r}\,/C\theta$ in the last integral yields

$$J(r) \le \frac{C}{A^2\sqrt{1-r}} \int_{\frac{\sqrt{1-r}}{C\pi}}^{\frac{1}{C\sqrt{1-r}}} e^{-x^2}\, dx \le \frac{C}{A^2\sqrt{1-r}} \int_0^{\infty} e^{-x^2}\, dx.$$

Thus there is a positive constant D such that

$$J(r) \le \frac{D}{\sqrt{1-r}} \tag{2.41}$$

for $0 \le r < 1$. Inequality (2.40) implies a similar estimate for I(r). Hence there is a positive constant E such that

$$\int_{-\pi}^{\pi} |S'(re^{i\theta})|\, d\theta \le \frac{E}{\sqrt{1-r}} \tag{2.42}$$

for $0 \le r < 1$.

Inequality (2.42) implies that

$$\int_0^1 \int_{-\pi}^{\pi} |S'(re^{i\theta})|\, (1-r)^{\alpha-1}\, d\theta\, dr \le E \int_0^1 (1-r)^{\alpha-3/2}\, dr.$$

If $\alpha > ½$ the last integral converges and thus S 0 $\mathcal{B}_\alpha$. Hence Theorem 2.14 implies that S 0 $\mathcal{F}_\alpha$ for $\alpha > ½$.

Next we verify that S ⌠ $\mathcal{F}_\alpha$ for $\alpha < ½$. From (a) and (c) of Lemma 2.17 we obtain

$$J(r) \ge \int_{1-r}^{\pi} \frac{1}{C^2\theta^2} \exp\left\{\frac{-2(1-r)}{A^2\theta^2}\right\} d\theta.$$

The substitution $x = \sqrt{2(1-r)}\,/\,A\theta$ in the last integral yields

$$J(r) \ge \frac{A}{C^2\sqrt{2(1-r)}} \int_{\frac{\sqrt{2(1-r)}}{A\pi}}^{\frac{1}{A}\sqrt{\frac{2}{1-r}}} e^{-x^2}\, dx.$$

Hence

$$\varliminf_{r\to 1-} \left\{\sqrt{1-r}\; J(r)\right\} \geq \frac{A}{C^2\sqrt{2}} \int_0^\infty e^{-x^2}\, dx.$$

Since

$$\int_{-\pi}^{\pi} |S'(re^{i\theta})|\, d\theta \geq 4\, J(r)$$

this implies that there is a positive constant F such that

$$\int_{-\pi}^{\pi} |S'(re^{i\theta})|\, d\theta \geq \frac{F}{\sqrt{1-r}} \tag{2.43}$$

for $0 \leq r < 1$. From (2.27) we see that (2.43) implies that $S \notin \mathcal{F}_\alpha$ when $\alpha < ½$.

Inequality (2.43) implies that $S \notin \mathcal{B}_{1/2}$. In Chapter 3 we obtain the stronger result that $S \notin \mathcal{F}_{1/2}$.

Next we obtain facts about $\mathcal{F}_1$ which relate to Hadamard products. Suppose that f and g are functions that are analytic in $\mathcal{D}$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ for $|z| < 1$. The Hadamard product of f and g is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \tag{2.44}$$

for $|z| < 1$. The series in (2.44) converges for $|z| < 1$ and hence $f * g$ is analytic in $\mathcal{D}$.

Two formulas are obtained for $f * g$. First, let $|z| < 1$ and choose ρ such that $|z| < \rho < 1$. Then

$$
\begin{aligned}
(f * g)(z) &= \sum_{n=0}^{\infty} a_n b_n z^n \\
&= \sum_{n=0}^{\infty} \left\{ \frac{1}{2\pi i} \int_{|w|=\rho} \frac{f(w)}{w^{n+1}} \, dw \right\} b_n z^n \\
&= \frac{1}{2\pi i} \int_{|w|=\rho} \frac{f(w)}{w} \left\{ \sum_{n=0}^{\infty} b_n \left(\frac{z}{w} \right)^n \right\} dw \\
&= \frac{1}{2\pi i} \int_{|w|=\rho} \frac{f(w)}{w} \, g\left(\frac{z}{w} \right) dw .
\end{aligned}
$$

Hence

$$
(f * g)(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\rho e^{i\theta}) \, g\left(\frac{z}{\rho} e^{-i\theta} \right) d\theta \tag{2.45}
$$

for $|z| < \rho < 1$.

For the second formula we assume that f 0 $\mathcal{F}_1$ and g is analytic in $\mathcal{D}$. Then there exists μ 0 $\mathcal{M}$ such that

$$
a_n = \int_T \overline{\zeta}^n \, d\mu(\zeta)
$$

for n = 0, 1, … . Hence

$$
\begin{aligned}
(f * g)(z) &= \sum_{n=0}^{\infty} \left\{ \int_T \overline{\zeta}^n \, d\mu(\zeta) \right\} b_n z^n \\
&= \int_T \left\{ \sum_{n=0}^{\infty} b_n \, \overline{\zeta}^n z^n \right\} d\mu(\zeta),
\end{aligned}
$$

which yields

$$
(f * g)(z) = \int_T g(\overline{\zeta} z) \, d\mu(\zeta) \tag{2.46}
$$

for $|z| < 1$.

Lemma 2.19 *If* $f \in \mathcal{F}_1^*$ *and* $g \in \mathcal{F}_1^*$ *then* $f * g \in \mathcal{F}_1^*$.

Proof: The function f belongs to $\mathcal{F}_1^*$ if and only if f is analytic in $\mathcal{D}$, $f(0) = 1$ and $\mathrm{Re}\, f(z) > ½$ for $|z| < 1$. This is an easy consequence of Theorem 1.1.

Suppose that $f \in \mathcal{F}_1^*$ and $g \in \mathcal{F}_1^*$. Then (2.46) holds for some $\mu \in \mathcal{M}^*$. Also $g(0) = 1$ and $\mathrm{Re}\, g(z) > ½$ for $|z| < 1$. Hence

$$(f * g)(0) = \int_{\mathrm{T}} g(0)\, d\mu(\zeta) = \int_{\mathrm{T}} d\mu(\zeta) = 1$$

and

$$\mathrm{Re}\, \{(f * g)(z)\} = \int_{\mathrm{T}} \mathrm{Re}\, g(\overline{\zeta} z)\, d\mu(\zeta) > \int_{\mathrm{T}} \frac{1}{2}\, d\mu(\zeta) = \frac{1}{2}$$

for $|z| < 1$. Therefore $f * g \in \mathcal{F}_1^*$.

Theorem 2.20 *If* $f \in \mathcal{F}_1$ *and* $g \in \mathcal{F}_1$ *then* $f * g \in \mathcal{F}_1$.

Proof: The hypothesis implies that f and g are linear combinations of four functions, each of which belongs to $\mathcal{F}_1^*$. Hence $f * g$ is a linear combination of sixteen functions, each of which is a Hadamard product of two functions in $\mathcal{F}_1^*$. Lemma 2.19 implies that $f * g \in \mathcal{F}_1$.

Theorem 2.21 *Let* f *be analytic in* $\mathcal{D}$. *Then* f *belongs to* $\mathcal{F}_1$ *if and if* $f * g \in H^\infty$ *for every* $g \in H^\infty$.

Proof: Suppose that $f \in \mathcal{F}_1$. Then there exists $\mu \in \mathcal{M}$ such that

$$f(z) = \int_{\mathrm{T}} \frac{1}{1 - \overline{\zeta} z}\, d\mu(\zeta) \qquad (|z| < 1). \tag{2.47}$$

Suppose that $g \in H^\infty$. Then (2.46) holds and hence

$$|(f * g)(z)| \le \int_{\mathrm{T}} \| g \|_{H^\infty}\, d|\mu|(\zeta) = \| g \|_{H^\infty} \| \mu \|$$

for $|z| < 1$. Therefore $f * g$ 0 H^∞. This proves one direction of the theorem. We also note that

$$\| f * g \|_{H^\infty} \le \| f \|_{F_1} \, \| g \|_{H^\infty} . \tag{2.48}$$

To prove the converse, suppose that f is analytic in $\mathcal{D}$ and $f * g$ 0 H^∞ for every g 0 H^∞. Let H_f be defined by $H_f(g) = f * g$ for g 0 H^∞. Our hypothesis is that H_f maps H^∞ into H^∞. Also H_f is linear.

We claim that $\mathcal{G} = \{(g, H_f\, g) : g$ 0 $H^\infty\}$ is closed in $H^\infty \times H^\infty$. To show this, suppose that g_n 0 H^∞ for n = 1, 2, … , g 0 H^∞, $g_n \to g$, $H_f\, g_n \to h$ and h 0 H^∞. We need to prove that $h = H_f\, g$. Suppose that $|z| < 1$ and let $r = |z|$. Choose ρ such that $r < \rho < 1$. Below we apply (2.45) with g replaced by $g - g_n$.

$$\begin{aligned} |(H_f\, g)(z) - h(z)| &\le |(H_f\, g)(z) - (H_f\, g_n)(z)| + |(H_f\, g_n)(z) - h(z)| \\ &= |H_f(g - g_n)(z)| + |(H_f\, g_n)(z) - h(z)| \\ &\le \left\{ \sup_{|z|=\rho} |f(z)| \right\} \| g - g_n \|_{H^\infty} + \| (H_f\, g_n) - h \|_{H^\infty} . \end{aligned}$$

Since $\| g - g_n \|_{H^\infty} \to 0$ and $\| (H_f\, g_n) - h \|_{H^\infty} \to 0$, this shows that $(H_f g)(z) = h(z)$.

Since H^4 is an F-space, H_f is linear and G is closed, the closed graph theorem implies that H_f is continuous. Thus H_f is a bounded linear operator on H^4. Therefore

$$M \equiv \sup \left\{ \frac{\| H_f\, g \|_{H^\infty}}{\| g \|_{H^\infty}} : \ g \,\square\, H^\infty, \ g \ne 0 \right\} < \infty \tag{2.49}$$

and $M = \| H_f \|$.

Recall that $\mathcal{C}$ is the Banach space of complex-valued functions which are defined and continuous on T and where the norm is defined by $\| g \|_C = \sup_{|\zeta|=1} |g(\zeta)|$ for g 0 $\mathcal{C}$. Let $\mathcal{A}$ denote the subspace of $\mathcal{C}$ consisting of functions that are analytic in $\mathcal{D}$ and continuous in $\overline{D}$. For $0 < r < 1$ define the function f_r by $f_r(z) = f(rz)$ for $|z| < \frac{1}{r}$. Then f_r is analytic in $\overline{D}$. Also define

$\varphi_r : \mathcal{A} \to |$ by

$$\varphi_r(g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_r(e^{i\theta})\, g(e^{-i\theta})\, d\theta \tag{2.50}$$

for g 0 $\mathcal{A}$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta})\, g(\rho e^{-i\theta})\, d\theta = \sum_{n=0}^{\infty} a_n b_n r^n \rho^n$$

for $0 < \rho < 1$. Letting $\rho \to 1$ we obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta})\, g(e^{-i\theta})\, d\theta = \sum_{n=0}^{\infty} a_n b_n r^n.$$

Hence

$$\varphi_r(g) = \sum_{n=0}^{\infty} a_n b_n r^n. \tag{2.51}$$

Since $\varphi_r(g) = (H_f g)(r)$, (2.49) yields

$$|\varphi_r(g)| \le M \, \| g \|_{H^\infty} \tag{2.52}$$

for $0 < r < 1$ and g 0 H^4.

A simple argument shows that φ_r is a continuous linear functional on $\mathcal{A}$. We also note that (2.52) implies that $\| \varphi_r \| \le M$ for $0 < r < 1$.

The Hahn-Banach theorem implies that φ_r can be extended to a continuous linear functional on $\mathcal{C}$ without increasing the norm. Let Φ_r denote such an extension. Then $\| \Phi_r \| \le M$ for $0 < r < 1$. Let $\{r_n\}$ ($n = 1, 2, \ldots$) be a sequence such that $0 < r_n < 1$ for all n and $r_n \to 1$ as $n \to 4$. The Banach-Alaoglu theorem implies that there is a subsequence of $\{r_n\}$, which we continue to call $\{r_n\}$, and Φ in the conjugate space $\mathcal{C}^*$ such that $\Phi_{r_n} \to \Phi$ in the weak* topology as $n \to 4$.

By the Riesz representation theorem there is a complex-valued Borel measure μ on $[-\pi, \pi]$ such that

$$\Phi(g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{-i\theta})\, d\mu(\theta) \tag{2.53}$$

for every g 0 $\mathcal{C}$. Let j be any nonnegative integer and define g_j by $g_j(e^{i\theta}) = e^{ji\theta}$. Then (2.51) yields $\varphi_{r_n}(g_j) = r_n^{\,j} a_j$ for n = 1, 2, … . Letting n → 4 in this equality and using (2.53) we obtain

$$a_j = \Phi(g_j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ji\theta}\, d\mu(\theta).$$

This can be expressed as $a_j = \int_T \overline{\zeta}^{\,j}\, d\upsilon(\zeta)$ where υ 0 $\mathcal{M}$. Since this relation holds for every nonnegative integer j we see that

$$f(z) = \sum_{j=0}^{\infty} a_j z^j = \int_T \frac{1}{1-\overline{\zeta}z}\, d\upsilon(\zeta)$$

for $|z| < 1$. This proves that f 0 $\mathcal{F}_1$.

NOTES

The facts about the gamma function given in Lemma 1 are in Andrews, Askey and Roy [1999]. Lemmas 3, 6 and 9 were proved by Brickman, Hallenbeck, MacGregor and Wilken [1973]. Equation (2.7) is a special case of a more general formula involving the hypergeometric functions given in Nehari [1952; see p. 206]. Lemma 6 was proved independently by Brannan, Clunie and Kirwan [1973]. Lemma 7, with the exception of the norm estimate, is in MacGregor [1987]. The exact norm estimate is in Hibschweiler and Nordgren [1996]. Theorems 8 and 10 are in MacGregor [1987] for $\alpha > 0$ and in Hibschweiler and MacGregor [1989] for $\alpha = 0$. Theorems 12, 14 and 18 were proved by Hallenbeck, MacGregor and Samotij [1996]. A proof of Lemma 13 is in Pommerenke [1962; see p. 262]. The argument in Theorem 16 showing that (2.34) implies f 0 $\mathcal{B}_\alpha$ is due to Protas [1973]. Theorem 21 was proved by Caveny [1966; see Theorem 3].

CHAPTER 3

Integral Means and the Hardy and Dirichlet Spaces

<u>Preamble</u>. This chapter focuses on relationships between the families F_a and other well-known families of functions analytic in D. The chapter begins with estimates on the integral mean for functions in F_a. Subsequently the membership of a function in F_a is related to its membership in a suitable Dirichlet space or Besov space.

We begin by introducing the notion of subordination. Littlewood's inequality is proved. It states that if two functions are related by subordination, then the integral mean is largest for the majorizing function. Set theoretic relations between F_a and the Hardy spaces H^p are obtained. One argument relies on a result of Hardy and Littlewood about fractional integrals and membership in H^p. Also asymptotic estimates are given for the growth of the integral means of a function in F_a as r ? 1–. This is a consequence of a comparable result of Hardy and Littlewood about Hardy spaces and fractional derivatives.

The Dirichlet spaces D_a are introduced and the membership of a function in D_a is related to its membership in $F_ß$ for suitable ß. This question is studied further for bounded functions and for inner functions. The membership of an inner function in F_a is shown to be equivalent to its membership in D_{1-a} and its membership in B_a, when $0 < a < 1$.

We begin with the notion of subordination. Let *f* and F denote two functions which are analytic in D. We say that *f* is subordinate to F in D provided that there exists a function f which is analytic in D and satisfies $| f(z) | < 1$ for $| z | < 1$ and $f(0) = 0$ and for which

$$f(z) = F(f(z)) \qquad (|z| < 1). \qquad (3.1)$$

If F is univalent in D, then *f* is subordinate to F if and only if *f*(D) d F(D) and *f*(0) = F(0). In this case, we see that f = F^{-1} B *f*. An example of subordination is given by the family P which consists of those functions that are subordinate to $F(z) = \dfrac{1+z}{1-z}$.

A fundamental result about subordination, Littlewood's inequality, is given next.

Theorem 3.1 *If f is subordinate to* F *in* D, p > 0 *and* $0 \le r < 1$, *then*

$$\int_{-p}^{p} | f(re^{i?}) |^p \, d? \le \int_{-p}^{p} | F(re^{i?}) |^p \, d\theta. \qquad (3.2)$$

Proof: Suppose that f is subordinate to F in D. Then (3.1) holds for a suitable function f. Let the functions u and U be defined by $u(z) = |f(z)|^p$ and $U(z) = |F(z)|^p$ for $|z| < 1$. Then u and U are subharmonic in D and $u(z) = U(f(z))$ for $|z| < 1$.

Since $f(0) = F(0)$, (3.2) holds when r = 0. Now suppose that $0 < r < 1$. Let V denote the function that is harmonic in $\{z : |z| < r\}$ and satisfies V(z) = U(z) for $|z| = r$. Since U is subharmonic this implies that $U(z) \le V(z)$ for $|z| \le r$. By Schwarz's Lemma, $|f(z)| \le |z|$, and thus $u(z) = U(f(z)) \le V(f(z))$ for $|z| \le r$. The function v = V B f is harmonic in $\{z: |z| < r\}$ and continuous in $\{z: |z| \le r\}$. Therefore

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})\, d\theta \le \frac{1}{2\pi}\int_{-\pi}^{\pi} v(re^{i\theta})\, d\theta = v(0) = V(0)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} V(re^{i\theta})\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(re^{i\theta})\, d\theta.$$

This proves (3.2). ÿ

For each function f analytic in D and for $0 \le r < 1$ and $0 < p < 8$ let

$$M_p(r, f) = \left\{ \frac{1}{2\pi}\int_{-\pi}^{\pi} | f(re^{i\theta}) |^p \, d\theta \right\}^{1/p}. \qquad (3.3)$$

For the next theorem we use the following known result.

Lemma 3.2 *Suppose that the function* u *is real-valued and nonnegative in a domain* Ω *and let* $v = u^p$ *where* $p > 0$*. If* u *is subharmonic in* Ω *and* $p \geq 1$ *then* v *is subharmonic in* Ω*. If* u *is superharmonic in* Ω *and* $0 < p < 1$ *then* v *is superharmonic in* Ω*.*

Theorem 3.3 *Suppose that* $f \in \mathcal{F}_\alpha$ *for some* $\alpha \geq 0$*. Then the following inequalities hold for* $0 \leq r < 1$ *where* A *denotes a constant having the form* $A = C\|f\|_{\mathcal{F}_\alpha}$ *and* C *is a positive constant depending only on* α *and* p*.*

(a) *If* $0 < \alpha \leq 1$ *and* $0 < p < 1/\alpha$ *then*

$$M_p^p(r, f) \leq A. \tag{3.4}$$

(b) *If* $\alpha = 0$ *then* (3.4) *holds for all* p $(0 < p < \infty)$.

(c) *If* $0 < \alpha \leq 1$ *and* $p = 1/\alpha$*, then*

$$M_p^p(r, f) \leq A \log \frac{2}{1-r}. \tag{3.5}$$

(d) *If* $\alpha > 0$, $p > 1/\alpha$ *and* $p \geq 1$*, then*

$$M_p^p(r, f) \leq \frac{A}{(1-r)^{\alpha p - 1}}. \tag{3.6}$$

(e) *If* $\alpha > 1$ *and* $0 < p < 1$*, then*

$$M_p^p(r, f) \leq \frac{A}{(1-r)^{(\alpha-1)p}}. \tag{3.7}$$

Proof: Suppose that $f \in \mathcal{F}_\alpha$ for some $\alpha > 0$. Then (1.1) holds for some $\mu \in \mathcal{M}$. To prove the inequalities it suffices to assume that $f \in \mathcal{F}_\alpha^*$. This is a consequence of the Jordan decomposition, (1.10) and the fact that there is a positive constant B depending only on p such that $(a+b)^p \leq B(a^p + b^p)$ for $a \geq 0$, $b \geq 0$ and $p > 0$.

Let $F_\alpha(z) = \dfrac{1}{(1-z)^\alpha}$ for $|z| < 1$. First consider the case $0 < \alpha \leq 1$. Then F_α is univalent in $\mathbb{D}$ and $F_\alpha(\mathbb{D})$ is convex. Since (1.1) can be expressed as

$f(z) = \int_T F_\alpha(\overline{\zeta}z)\, d\mu(\zeta)$, the convexity of $F_a(D)$ and the fact that μ is a probability measure imply that $f(D)$ d $F_a(D)$. Also $f(0) = F_a(0) = 1$ and hence the univalence of F_a implies that f is subordinate to F_a. Theorem 3.1 yields

$$M_p^p(r, f) \le M_p^p(r, F_\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|1 - re^{i\theta}|^{\alpha p}}\, d\theta.$$

If $0 < p \le 1/a$ then (2.26) yields (3.4) and (3.5).

Since the function z ?? $\log \dfrac{1}{1 - z}$ is univalent and convex in D and belongs to H^p for every p ($0 < p < 8$), we find that when $a = 0$, (3.4) holds for every p.

From (1.1) and μ 0 M* we obtain

$$|f(z)| \le \int_T \frac{1}{|1 - \overline{\zeta}z|^\alpha}\, d\mu(\zeta) \qquad (|z| < 1). \tag{3.8}$$

Suppose that $p \ge 1$. Then (3.8), the continuous form of Minkowski's inequality and periodicity yield

$$M_p(r, f) \le M_p(r, F_a) \tag{3.9}$$

for $0 \le r < 1$. Hence (2.26) yields (3.6).

Next assume that $a > 1$ and $0 < p < 1$. First suppose that $a \ge 2$. Then (3.8) implies that

$$\begin{aligned} |f(z)| &\le \frac{1}{(1 - |z|)^{\alpha-2}} \int_T \frac{1}{|1 - \overline{\zeta}z|^2}\, d\mu(\zeta) \\ &\le \frac{1}{(1 - |z|)^{\alpha-1}} \int_T \frac{1 - |z|^2}{|1 - \overline{\zeta}z|^2}\, d\mu(\zeta) \\ &= \frac{1}{(1 - |z|)^{\alpha-1}} \int_T \operatorname{Re}\left(\frac{1 + \overline{\zeta}z}{1 - \overline{\zeta}z}\right) d\mu(\zeta). \end{aligned}$$

Hence

$$|f(z)| \le \frac{u(z)}{(1 - |z|)^{\alpha-1}} \tag{3.10}$$

where u is a positive harmonic function. Since $0 < p < 1$ Lemma 3.2 implies that u^p is superharmonic in $\mathbb{D}$ and therefore

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u^p(re^{i\theta})\, d\theta \le u^p(0) = 1$$

for $0 \le r < 1$. Hence (3.10) gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p\, d\theta \le \frac{1}{(1-r)^{(\alpha-1)p}}. \tag{3.11}$$

This proves (3.7) when $a \ge 2$.

Finally assume that $0 < p < 1$ and $1 < a < 2$. Theorem 2.8 implies that $g = f' \in \mathcal{F}_{a+1}$. Since $a + 1 > 2$, the previous result applies to g and hence

$$M_p(r, f') \le \frac{B}{(1-r)^{\alpha}} \tag{3.12}$$

for $0 \le r < 1$ and for a suitable positive constant B. Because $a > 1$, (3.12) is known to imply

$$M_p(r, f) \le \frac{C}{(1-r)^{\alpha-1}} \tag{3.13}$$

for a suitable constant C. This proves (3.7) in the case $1 < a < 2$.

The arguments given here show that the constant A in (3.4), (3.5), (3.6) and (3.7) has the form we claimed. □

The sharpness of (3.5) and (3.6) follows by letting $f = F_a$ and appealing to Lemma 2.13. In Theorem 3.8 it is shown that if $f \in \mathcal{F}_a$, $a > 1$ and $0 < p < 1$ then $\lim_{r\to 1-} (1-r)^{(\alpha-1)p} M_p^p(r, f) = 0$ and so no function can exhibit sharpness for (3.7).

Next we obtain relations between $\mathcal{F}_a$ and H^p. One argument uses the following result of Hardy and Littlewood.

Theorem 3.4 *Suppose that* $0 < q < t < 8$, $ß = \frac{1}{q} - \frac{1}{t}$ *and* $?_n = \frac{n!}{G(n + 1 + ß)}$ *for* $n = 0, 1, \ldots$. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n \ (|z| < 1)$ *and let*

$$g(z) = \sum_{n=0}^{\infty} b_n z^n \ (|z| < 1)$$ *where* $b_n = ?_n a_n$ *for* $n = 0, 1, \ldots$. *If* f 0 H^q *then* g 0 H^t.

Theorem 3.5 F_0 d H^p *for* $0 < p < 8$. *If* $0 < a \le 1$ *then* F_a d H^p *for* $0 < p < 1/a$. *If* $0 < p \le 1$ *then* H^p d $F_{1/p}$. *If* $p > 1$ *then* H^p d F_1.

Proof: The first two statements are consequences of Theorem 3.3. Earlier we noted that H^1 d F_1. Next suppose that $0 < p < 1$ and f 0 H^p and let $f(z) = \sum_{n=0}^{\infty} a_n z^n \ (|z| < 1)$. Applying Theorem 3.4 with $q = p$ and $t = 1$ we obtain g 0 H^1 where $g(z) = \sum_{n=0}^{\infty} b_n z^n$, $b_n = \gamma_n a_n$ and $\gamma_n = \frac{n!}{\Gamma(n + \frac{1}{p})}$.

Equation (2.5) yields $a_n = A_n(1/p)\,\Gamma(1/p)\,b_n$. Because g 0 F_1, this coefficient relation and the remark after relation (1.18) imply that f 0 $F_{1/p}$.

The last assertion follows since $p > 1$ implies that H^p d H^1 d F_1. ÿ

Simple examples show that the relations in Theorem 3.5 are precise. In Chapter 5 we give an example which implies that if $a > 1$ there is a function in F_a which belongs to no Hardy space.

The following example relates to Theorem 3.5.

Proposition 3.6 *For* $a > 0$ *and* $ß > 0$ *let*

$$f(z) = \frac{1}{(1-z)^{\alpha} \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{\beta}} \qquad (|z| < 1).$$

If $ß > a$ *then* f 0 $H^{1/a}$ *and if* $ß = a$ *then* f ó $H^{1/a}$. *Also* f 0 F_a *for every* a *and* $ß$.

Proof: Let $\alpha > 0$ and $\beta > 0$. The function $z \mapsto \dfrac{1}{\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{\beta}}$ is bounded in D. Also the function $z \mapsto \dfrac{1}{(1-z)^{\alpha}}$ belongs to H^p for $p < \dfrac{1}{\alpha}$, and hence $f \in H^p$ for $p < \dfrac{1}{\alpha}$. Because $f \in H^p$ for some p it follows that $f \in H^q$ $(q > p)$ if and only if $f(e^{i\theta}) \in L^q([-\pi, \pi])$.

There are positive constants A and B (depending only on α and β) such that

$$\frac{A}{|\theta|^{\alpha}\left(\log\frac{1}{|\theta|}\right)^{\beta}} \le |f(e^{i\theta})| \le \frac{B}{|\theta|^{\alpha}\left(\log\frac{1}{|\theta|}\right)^{\beta}} \tag{3.14}$$

for $0 < |\theta| \le 1/2$. Also f extends continuously to $\partial D \setminus \{1\}$. Since $\displaystyle\int_0^{1/2} \frac{1}{\theta(\log\frac{1}{\theta})^{\gamma}}\, d\theta < \infty$ for $\gamma > 1$, we conclude from the right-hand inequality in (3.14) that $f(e^{i\theta}) \in L^{1/\alpha}([-\pi, \pi])$ when $\beta > \alpha$. Therefore $f \in H^{1/\alpha}$ when $\beta > \alpha$. Since $\displaystyle\int_0^{1/2} \frac{1}{\theta(\log\frac{1}{\theta})}\, d\theta = \infty$, the left-hand inequality in (3.14) implies that $f(e^{i\theta}) \notin L^{1/\alpha}([-\pi, \pi])$ when $\beta = \alpha$. Therefore $f \notin H^{1/\alpha}$ when $\beta = \alpha$.

It remains to show that $f \in F_{\alpha}$. First consider the case $\alpha = 1$. If $\beta > 1$ then as shown above $f \in H^1$ and hence $f \in F_1$. Next suppose that $0 < \beta \le 1$. Let

$$g(z) = \frac{1}{(1-z)\,\frac{1}{z}\log\frac{1}{1-z}} \qquad (|z| < 1)$$

and $h = \dfrac{1}{g}$. Then $h(z) = \displaystyle\int_0^1 \frac{1-z}{1-tz}\, dt$ and since $\text{Re}\left(\dfrac{1-z}{1-tz}\right) > 0$ for $0 \le t \le 1$ and $|z| < 1$ we have $\text{Re}\, h(z) > 0$. Let $k(z) = (1-z)^{1-\beta} h^{\beta}(z)$. Since $0 < \beta \le 1$, $\text{Re}\,(1-z) > 0$ and $\text{Re}\, h(z) > 0$ it follows that $\text{Re}\, k(z) > 0$. Because $f = 1/k$ we obtain $\text{Re}\, f(z) > 0$ for $|z| < 1$. The Riesz-Herglotz formula yields $f \in F_1$.

Next we consider the case a > 1. Let ? = a – 1. Then

$$f(z) = \frac{1}{(1-z)^{\gamma}} \left\{ \frac{1}{(1-z)} \frac{1}{\left(\frac{1}{z} \log \frac{1}{1-z}\right)^{\beta}} \right\}.$$

The second function in this product is in F_1, by the previous case. Since the first function is in $\mathsf{F}_?$, Theorem 2.7 implies that f 0 F_a.

Finally, assume that 0 < a < 1. Then

$$f'(z) = \frac{\beta}{z}\left(\ell(z) + m(z)\right) + n(z) \tag{3.15}$$

where

$$\ell(z) = -\frac{1}{(1-z)^{\alpha+1}\left[\frac{1}{z}\log\frac{1}{1-z}\right]^{\beta+1}},$$

$$m(z) = \frac{1}{(1-z)^{\alpha}\left[\frac{1}{z}\log\frac{1}{1-z}\right]^{\beta}}$$

and

$$n(z) = \frac{\alpha}{(1-z)^{\alpha+1}\left[\frac{1}{z}\log\frac{1}{1-z}\right]^{\beta}}.$$

The previous case yields R 0 F_{a+1}. Since $m(z) = \frac{1}{(1-z)^{\alpha}}\, p(z)$ and p 0 H^8, Theorem 2.7 yields m 0 F_{a+1}. Because R + m 0 F_{a+1} and R + m vanishes at 0 it follows easily that the function z ?? $\frac{\beta}{z}\left[\ell(z) + m(z)\right]$ also belongs to F_{a+1}. Also n 0 F_{a+1} by the previous case. Therefore (3.15) implies that f' 0 F_{a+1}. Theorem 2.8 yields f 0 F_a. ÿ

For $\beta > 0$, let the operators I_β and D_β be defined on H as follows. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$, let

$$(I_\beta f)(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n-\beta+1)}{n!} a_n z^n \quad \text{and}$$

$$(D_\beta f)(z) = \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n-\beta+1)} a_n z^n .$$

Then $(D_\beta f)(z) = z^\beta f^{(\beta)}(z)$ where $f^{(\beta)}$ denotes the fractional derivative of f. We shall use the following theorem of Hardy and Littlewood.

Theorem 3.7 *If* $f \in H^p$ *for some* $p > 0$ *then* $\lim_{r\to 1-} (1-r)^\beta M_p(r, D_\beta f) = 0$ *for every* $\beta > 0$.

Theorem 3.8 *If* $0 < p < 1$, $a > 1$ *and* $f \in \mathcal{F}_a$, *then* $\lim_{r\to 1-} (1-r)^{a-1} M_p(r, f) = 0$.

Proof: Suppose that $a > 1$ and $f \in \mathcal{F}_a$ and let $\beta = a-1$. Then (1.18) and (2.5) imply that

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+\beta+1)}{\Gamma(\beta+1)\, n!} \int_T \overline{\zeta}^n \, d\mu(\zeta) \right) z^n \quad (|z|<1) \tag{3.16}$$

for some $\mu \in \mathcal{M}$. Let $g = I_\beta f$. Then

$$g(z) = \sum_{n=0}^{\infty} \left(\frac{\Gamma(n-\beta+1)\, \Gamma(n+\beta+1)}{\Gamma(\beta+1)\, (n!)^2} \int_T \overline{\zeta}^n \, d\mu(\zeta) \right) z^n \quad (|z|<1). \tag{3.17}$$

For γ real, apply (2.4) with $z = n+\gamma$ and $z = n+1$. Division of the two results yields

$$\frac{\Gamma(n+\gamma)}{n!} = n^{\gamma-1} \left[1 + O\left(\frac{1}{n}\right) \right] \tag{3.18}$$

as $n \to \infty$. Two applications of (3.18) yield

$$\frac{\Gamma(n-\beta+1)\ \Gamma(n+\beta+1)}{(n!)^2} = 1 + b_n \tag{3.19}$$

as $n \to \infty$, where the sequence $\{b_n\}$ satisfies $b_n = O\left(\frac{1}{n}\right)$. From (3.17) and (3.19) we see that

$$g = g_1 + g_2 \tag{3.20}$$

where $g_1(z) = \frac{1}{\Gamma(\beta+1)} \int_T \frac{1}{1-\overline{\zeta}z}\, d\mu(\zeta)$, $g_2(z) = \frac{1}{\Gamma(\beta+1)} \sum_{n=0}^{\infty} c_n z^n$ and $c_n = b_n \int_T \overline{\zeta}^n\, d\mu(\zeta)$. Hence $\sum_{n=0}^{\infty} |c_n|^2 < \infty$ and therefore $g_2 \in H^2 \subset \mathcal{F}_1$. Also $g_1 \in \mathcal{F}_1$ and thus (3.20) implies that $g \in \mathcal{F}_1$. Hence (3.4) yields $g \in H^p$ for $0 < p < 1$. Theorem 3.7 now implies that

$$\lim_{r\to 1-} (1-r)^{\beta} M_p(r, D_\beta g) = 0.$$

Since $D_\beta g = f$ and $\beta = \alpha - 1$ this proves $\lim_{r\to 1-} (1-r)^{\alpha-1} M_p(r, f) = 0$. ∎

If the function f is analytic in $\mathbb{D}$ and $f \neq 0$ we let

$$M_0(r, f) = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\, d\theta\right] \tag{3.21}$$

for $0 < r < 1$. Then the integral in (3.21) exists, $M_0(r, f) \le M_p(r, f)$ for $p > 0$ and

$$\lim_{p\to 0+} M_p(r, f) = M_0(r, f).$$

The following result is a consequence of Theorem 3.8.

Corollary 3.9 *If* $f \in \mathcal{F}_\alpha$ *for some* $\alpha > 1$ *and* $f \neq 0$ *then*

$$\lim_{r\to 1-} (1-r)^{\alpha-1} M_0(r, f) = 0.$$

In Chapter 5 it is shown that Theorem 3.8 and Corollary 3.9 are sharp in a precise way.

Next we consider Dirichlet spaces. For each real number α, let $\mathcal{D}_\alpha$ denote the set of functions f that are analytic in $\mathcal{D}$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($|z| < 1$) and

$$\sum_{n=1}^{\infty} n^{\alpha} |a_n|^2 < \infty. \tag{3.22}$$

The space $\mathcal{D}_\alpha$ is a Banach space with respect to the norm

$$\| f \|_{\mathcal{D}_\alpha} \equiv |f(0)| + \left(\sum_{n=1}^{\infty} n^{\alpha} |a_n|^2 \right)^{1/2}. \tag{3.23}$$

The cases $\alpha = 0$ and $\alpha = 1$ are of special importance. Note that $\mathcal{D}_0 = H^2$. Let $A(\Omega)$ denote the Lebesgue area measure of the measurable set $\Omega \subset \mathbb{C}$. Since $dA = r\, d\theta\, dr$, the power series for f yields

$$\int_{|z| < r} |f'(z)|^2 \, dA(z) = \pi \sum_{n=1}^{\infty} n |a_n|^2 r^{2n}$$

for $0 < r < 1$. This is the area of the image of $\{z: |z| < r\}$ under the map $z \longmapsto f(z)$, where we count multiplicities of the covered regions. Hence $f \in \mathcal{D}_1$ provided that the area of the map $z \longmapsto f(z)$ is bounded for $0 < r < 1$. In particular, if f is univalent in $\mathcal{D}$ then $f \in \mathcal{D}_1$ if and only if the area of $f(\mathcal{D})$ is finite.

If $\alpha < \beta$ then $\mathcal{D}_\beta \subset \mathcal{D}_\alpha$. Also, if $\alpha > 1$ and $f \in \mathcal{D}_\alpha$ then $\sum_{n=1}^{\infty} |a_n| < \infty$ and so f extends continuously to $\overline{D}$. In particular, $\mathcal{D}_\alpha \subset H^\infty$ for $\alpha > 1$.

Theorem 3.10 *If* $\alpha \le 2$ *then* $\mathcal{D}_\alpha \subset \mathcal{F}_{1-\alpha/2}$.

Proof: Suppose that $f \in \mathcal{D}_\alpha$ for some $\alpha \le 2$, and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($|z| < 1$). Define g by $g(z) = \sum_{n=1}^{\infty} b_n z^n$ ($|z| < 1$) where

$b_n = n^{\alpha/2} a_n$ for n = 1,2, … . Then (3.22) implies $\sum_{n=1}^{\infty} |b_n|^2 < \infty$. Thus g 0 H^2 and hence g 0 $\mathcal{F}_1$. It follows that there exists μ 0 $\mathcal{M}$ such that $b_n = \int_T \overline{\zeta}^n \, d\mu(\zeta)$ for n = 1, 2, … . In the case α = 2 this yields $a_n = \frac{1}{n} \int_T \overline{\zeta}^n \, d\mu(\zeta)$, that is, f 0 $\mathcal{F}_0$. Suppose that α < 2 and let β = 1–α/2. Then $a_n = n^{\beta-1} \int_T \overline{\zeta}^n \, d\mu(\zeta)$, for n = 1, 2, … . The relation (2.9) yields
$n^{\beta-1} = A_n(\beta)\left[\Gamma(\beta) + c_n(\beta)\right]$ for n = 1, 2, … where $|c_n(\beta)| = O\left(\frac{1}{n}\right)$ as $n \to \infty$.
Hence

$$a_n = \Gamma(\beta) A_n(\beta) \int_T \overline{\zeta}^n \, d\mu(\zeta) + c_n(\beta) \, A_n(\beta) \int_T \overline{\zeta}^n \, d\mu(\zeta)$$

for n = 1, 2, … . Since $|c_n(\beta)| = O\left(\frac{1}{n}\right)$ the function $h(z) = \sum_{n=1}^{\infty} d_n z^n$ where $d_n = c_n(\beta) \int_T \overline{\zeta}^n \, d\mu(\zeta)$ belongs to H^2 and hence there exists υ 0 $\mathcal{M}$ such that $d_n = \int_T \overline{\zeta}^n \, d\upsilon(\zeta)$ for n = 1, 2, … . Therefore $a_n = A_n(\beta) \int_T \overline{\zeta}^n \, d\lambda(\zeta)$ for n = 1, 2, … where $\lambda = \Gamma(\beta)\mu + \upsilon$. The argument shows that $\sum_{n=1}^{\infty} a_n z^n$ 0 $\mathcal{F}_\beta$ and hence f 0 $\mathcal{F}_{1-\alpha/2}$. □

It follows from our argument that if $\alpha \leq 2$ then each function in $\mathcal{D}_\alpha$ is represented in $\mathcal{F}_{1-\alpha/2}$ by a measure μ such that $d\mu(\zeta) = F(e^{i\theta})\, d\theta$ ($\zeta = e^{i\theta}$) where F 0 L^2 ([–π, π]).

Theorem 3.11 *If* $\alpha \geq 0$ *and* $\beta < 1 - 2\alpha$ *then* $\mathcal{F}_\alpha$ δ $\mathcal{D}_\beta$.

Proof: Suppose that $\alpha \geq 0$ and f 0 $\mathcal{F}_\alpha$ where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($|z| < 1$). The asymptotic expansion for $A_n(\alpha)$ shows that there is a positive constant A such that $|a_n| \leq A \, n^{\alpha-1}$ for n = 1, 2, … . This yields

$\sum_{n=1}^{\infty} n^{\beta} \, |a_n|^2 \le A^2 \sum_{n=1}^{\infty} n^{2\alpha+\beta-2}$ for each $\beta \ge 0$. If $\beta < 1 - 2\alpha$ the last sum is finite and therefore $f \in \mathcal{D}_\beta$.

The arguments given for Theorems 3.10 and 3.11 provide suitable comparisons of the norms. The theorems are sharp in the following ways. There is a function $f \in \mathcal{D}_\alpha$ such that $f \notin \mathcal{F}_\beta$ for every $\beta < 1 - \alpha/2$. An example is given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $a_n = \dfrac{\log 2}{(\log n)\, n^{\alpha/2}}$ for $n = 2^k$ ($k = 1, 2, \ldots$) and otherwise $a_n = 0$. Also $\mathcal{F}_\alpha \not\subset \mathcal{D}_{1-2\alpha}$ follows by considering the function $z \mapsto \dfrac{1}{(1-z)^{\alpha}}$.

Next we improve Theorem 3.11 for bounded functions.

Theorem 3.12 *If* $\alpha \ge 0$ *then* $\mathcal{F}_\alpha \cap H^\infty \subset \mathcal{D}_{1-\alpha}$.

Proof: Suppose that $\alpha \ge 0$ and $f \in \mathcal{F}_\alpha \cap H^\infty$. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (|z| < 1)$$

and $g(z) = \sum_{n=1}^{\infty} \dfrac{a_n}{n^{\alpha-1}} z^n$ ($|z| < 1$). Since $f \in \mathcal{F}_\alpha$ earlier arguments (such as in the proof of Theorem 3.10) imply that $g \in \mathcal{F}_1$. Hence there exists $\mu \in \mathcal{M}$ such that

$$g(z) = \int_T \frac{1}{1 - \overline{\zeta} z} \, d\mu(\zeta) \qquad (|z| < 1). \tag{3.24}$$

Let $h(z) = \overline{f(\overline{z})}$ for $|z| < 1$. Since $f \in H^\infty$ it follows that $h \in H^\infty$. Let k denote the Hadamard product of h and g. Theorem 2.21 implies that $k \in H^\infty$. Since

$h(z) = \sum_{n=0}^{\infty} \overline{a}_n z^n$ ($|z| < 1$),

$$k(z) = \sum_{n=1}^{\infty} n^{1-\alpha} \, |a_n|^2 z^n \quad (|z| < 1). \tag{3.25}$$

Because k is bounded, it follows that $\sum_{n=1}^{\infty} n^{1-\alpha} \, |a_n|^2 < \infty$. Therefore $f \in \mathcal{D}_{1-\alpha}$.

From our earlier observations about $\mathcal{D}_1$, Theorem 3.12 can be restated in the following way when $\alpha = 0$.

Corollary 3.13 For $0 < r < 1$ let $A(r)$ denote the area of $f(\{z: |z| < r\})$ counting multiple coverings. If $f \in \mathcal{F}_0 \cap H^\infty$ then $\lim_{r \to 1-} A(r) < \infty$.

We shall determine those inner functions which belong to $\mathcal{F}_0$. A function f is called an inner function provided that f is analytic in $\mathcal{D}$, $|f(z)| \le 1$ for $|z| < 1$ and $|F(\theta)| = 1$ for almost all θ in $[-\pi, \pi]$, where $F(\theta) \equiv \lim_{r \to 1-} f(re^{i\theta})$. Each finite or infinite Blaschke product is an inner function and so is the function S where

$$S(z) = \exp\left[-\frac{1+z}{1-z}\right] \text{ for } |z| < 1.$$

Recall that $A(E)$ denotes the Lebesgue area measure of the measurable set $E \subset \mathbb{C}$.

Lemma 3.14 *Let f be analytic in $\mathcal{D}$ and let*

$$E = \{w \in \mathbb{C} : f - w \text{ has an infinite number of zeros in } \mathcal{D}\}.$$

If E is measurable and $A(E) > 0$, *then* $\int_{\mathcal{D}} |f'(z)|^2 \, dA(z) = \infty$.

Proof: For $w \in \mathbb{C}$, let $N_f(w)$ denote the number of zeros, counting multiplicities, of $f - w$ in $\mathcal{D}$. The change of variable formula gives

$$\int_{\mathcal{D}} |f'(z)|^2 \, dA(z) = \int_{f(\mathcal{D})} N_f(w) \, dA(w). \qquad (3.26)$$

By hypothesis, $N_f(w) = \infty$ for all $w \in E$. Hence (3.26) implies that

$$\int_{\mathcal{D}} |f'(z)|^2 \, dA(z) \ge \int_{E} N_f(w) \, dA(w) = \infty.$$

We shall use a weaker form of the following result of Frostman.

Lemma 3.15 *Suppose that f is an inner function which is neither constant nor a finite Blaschke product. Then the set*

$$\{ w \in \mathbb{D} : f(z) - w = 0 \text{ for finitely many } z \in \mathbb{D}\}$$

is of logarithmic capacity zero.

Theorem 3.16 *If f is an inner function which is neither constant nor a finite Blaschke product, then* $\int_{D} |f'(z)|^2 \, dA(z) = \infty$.

Proof: This follows from Lemmas 3.14 and 3.15 since every set of logarithmic capacity zero has Lebesgue measure zero, that is, the set E in Lemma 3.14 satisfies $A(E) = \pi$.

Theorem 3.17 *An inner function belongs to $\mathcal{D}_0$ if and only if it is a constant of modulus 1 or a finite Blaschke product.*

Proof: It is easy to show that any function analytic in $\overline{D}$ belongs to $\mathcal{D}_0$. In particular, every constant function and every finite Blaschke product belong to $\mathcal{D}_0$.

Conversely, if f is an inner function and $f \in \mathcal{D}_0$ then Theorem 3.16 and Corollary 3.13 imply that f is a constant of modulus 1 or f is a finite Blaschke product.

Next we obtain a relation between membership in a Dirichlet space and membership in a Besov space for inner functions.

Let $\mathcal{B}$ denote the set of functions f that are analytic in $\mathbb{D}$ and satisfy $|f(z)| \le 1$ for $|z| < 1$. For $f \in \mathcal{B}$ and $0 \le r < 1$, let

$$E(r) = 1 - \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \, d\theta. \tag{3.27}$$

Then $E(r) \ge 0$ and E is nonincreasing on [0,1).

Lemma 3.18 *If $f \in \mathcal{B}$, $0 < \alpha < 1$ and* $\int_0^1 E(r)(1-r)^{\alpha-2} \, dr < \infty$, *then $f \in \mathcal{B}_\alpha$.*

Proof: Suppose that $f \in \mathcal{B}$ and $0 < \alpha < 1$. Then $|f'(z)| \le \dfrac{1 - |f(z)|^2}{1 - |z|^2}$ for $|z| < 1$. Therefore

$$\int_0^1 \int_{-\pi}^{\pi} |f'(re^{i\theta})|\,(1-r)^{\alpha-1}\ d\theta\ dr \le \int_0^1 \int_{-\pi}^{\pi} (1-|f(re^{i\theta})|^2)\ (1-r)^{\alpha-2}\ d\theta\ dr$$

$$= 2\pi \int_0^1 E(r)\ (1-r)^{\alpha-2}\ dr.$$

The hypothesis $\int_0^1 E(r)(1-r)^{\alpha-2}\,dr < \infty$ and the definition (2.21) imply that f 0 $\mathcal{B}_\alpha$.

Suppose that f is an inner function. Then $\lim_{r\to 1-} E(r) = 0$. This is a consequence of the following facts. First, if $F(\theta) = \lim_{r\to 1-} f(re^{i\theta})$ then $|F(\theta)| = 1$ for almost all θ and hence $\int_{-\pi}^{\pi} |F(\theta)|^2\ d\theta = 2\pi$. Also, since f 0 H^2,

$$\lim_{r\to 1-} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2\ d\theta = \int_{-\pi}^{\pi} |F(\theta)|^2\ d\theta.$$

Lemma 3.19 *Suppose that f is an inner function. Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(|z|<1)$ *and for* $k = 0, 1, \ldots$ *let* $R_k = \sum_{n=k}^{\infty} |a_n|^2$. *Then*
$E(r) \le 4(1-r) \sum_{1 \le n \le \frac{1}{1-r}} R_n$ *for* $0 \le r < 1$.

Proof: Suppose that $0 \le r < 1$, and let m be the greatest integer in $\frac{1}{1-r}$. By assumption f is an inner function and hence

$$1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(\theta)|^2\ d\theta = \sum_{n=0}^{\infty} |a_n|^2 .$$

Therefore

$$E(r) = \sum_{n=0}^{\infty} |a_n|^2 - \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

$$= \sum_{n=1}^{m} |a_n|^2 (1 - r^{2n}) + \sum_{n=m+1}^{\infty} |a_n|^2 (1 - r^{2n})$$

$$\leq (1 - r^2) \sum_{n=1}^{m} n |a_n|^2 + \sum_{n=m+1}^{\infty} |a_n|^2$$

$$= (1 - r^2) \sum_{n=1}^{m} n(R_n - R_{n+1}) + R_{m+1}$$

$$= (1 - r^2) \left[\sum_{n=1}^{m} R_n - m R_{m+1} \right] + R_{m+1}$$

$$\leq 2(1 - r) \sum_{n=1}^{m} R_n + [1 - m(1 - r^2)] R_{m+1}.$$

The definition of m yields $1 - \frac{1}{m} \leq r < 1 - \frac{1}{m+1}$. If $r = 1 - \frac{1}{m}$ then

$$1 - m(1 - r^2) = -1 + \frac{1}{m} \leq 0.$$

It follows that $E(1 - \frac{1}{m}) \leq \frac{2}{m} \sum_{n=1}^{m} R_n$. Because E is nonincreasing this implies that $E(r) \leq \frac{2}{m} \sum_{n=1}^{m} R_n$. Also $r < 1 - \frac{1}{m+1}$ and $\frac{1}{m} \leq \frac{2}{m+1}$ and hence

$$E(r) \leq 4(1 - r) \sum_{n=1}^{m} R_n \leq 4(1 - r) \sum_{1 \leq n \leq \frac{1}{1-r}} R_n .$$

Lemma 3.20 *Suppose that* $0 < \alpha < 1$ *and* $\sum_{n=1}^{\infty} n^{1-\alpha} |a_n|^2 < \infty$.

For k = 0, 1, … *let* $R_k = \sum_{n=k}^{\infty} |a_n|^2$. *Then there is a positive constant* A *depending only on* α *such that*

$$\int_0^1 \Big\{ \sum_{1 \le n \le \frac{1}{1-r}} R_n \Big\} (1-r)^{\alpha-1}\, dr \le A \sum_{n=1}^{\infty} n^{1-\alpha} |a_n|^2 .$$

Proof: The assumptions $0 < \alpha < 1$ and $\sum_{n=1}^{\infty} n^{1-\alpha} |a_n|^2 < \infty$ imply that $\sum_{n=1}^{\infty} |a_n|^2 < \infty$, and hence $R_k < \infty$ for every k. There is a positive constant B depending only on α such that $\sum_{k=1}^{n} \frac{1}{k^{\alpha}} \le B n^{1-\alpha}$ for n = 1, 2, … . Thus

$$\int_0^1 \Big\{ \sum_{1 \le n \le \frac{1}{1-r}} R_n \Big\} (1-r)^{\alpha-1}\, dr = \sum_{n=1}^{\infty} \Big\{ R_n \int_{1-\frac{1}{n}}^{1} (1-r)^{\alpha-1}\, dr \Big\}$$

$$= \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{R_n}{n^{\alpha}} = \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \sum_{k \ge n} |a_k|^2$$

$$= \frac{1}{\alpha} \sum_{n=1}^{\infty} |a_n|^2 \Big\{ \sum_{k=1}^{n} \frac{1}{k^{\alpha}} \Big\} \le \frac{B}{\alpha} \sum_{n=1}^{\infty} n^{1-\alpha} |a_n|^2 .$$

Theorem 3.21 *Suppose that f is an inner function and* $0 < \alpha < 1$. *If f* 0 $\mathcal{D}_{1-\alpha}$ *then f* 0 $\mathcal{B}_{\alpha}$.

Proof: Assume that $0 < \alpha < 1$, *f* is an inner function and *f* 0 $\mathcal{D}_{1-\alpha}$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n \ (|z| < 1)$ then by assumption $\sum_{n=1}^{\infty} n^{1-\alpha} |a_n|^2 < \infty$. We obtain the conclusion *f* 0 $\mathcal{B}_{\alpha}$ by successively applying Lemmas 3.20, 3.19 and 3.18.

Theorem 3.22 *Suppose that f is an inner function and* $0 < \alpha < 1$. *The following are equivalent.*

(a) $f \in \mathcal{F}_\alpha$.

(b) $f \in \mathcal{D}_{1-\alpha}$.

(c) $f \in \mathcal{B}_\alpha$.

Proof: This theorem is a consequence of Theorems 2.14, 3.12 and 3.21.

Theorem 3.22 completes the information about $S(z) = \exp\left[-\frac{1+z}{1-z}\right]$ obtained in Theorem 2.18. As we noted in Chapter 2, $S \notin \mathcal{B}_{1/2}$. Now we see that this fact and Theorem 3.22 yield $S \notin \mathcal{F}_{1/2}$.

NOTES

Theorem 1 was proved by Littlewood [1925]. The argument given here is due to Riesz [1925]. Duren [1983; see Chapter 6] contains a variety of facts about subordination. Theorems 3, 8 and 9 were proved by Hallenbeck and MacGregor [1993a]. Theorems 4 and 7 are due to Hardy and Littlewood [1932]. Theorem 5 is in MacGregor [1987]. The result used in the proof of Proposition 6, namely, if $f \in H^p$ and $q > p$, then $f \in H^q$ if and only if $f(e^{i\theta}) \in L^q([-\pi, \pi])$, appears in Duren [1970; p. 28]. Hallenbeck, MacGregor and Samotij [1996] proved Theorems 10, 11, 12, 17 and 22. Equation (3.26) is known and a more general change of variable formula is given in Cowen and MacCluer [1994; see p. 36]. Lemma 15 was proved by Frostman [1935]. Theorem 16 was proved by Newman and H.S. Shapiro [1962] using ideas about dual extremal problems. Later Erdös, H.S. Shapiro and Shields [1965] gave a proof based on results about the dimension of subspaces of ℓ^2. Two additional arguments for Theorem 16, one of which is a direct geometric proof, have been obtained by K. Samotij. Lemmas 18, 19 and 20 are due to Ahern [1979], who proved that if f is an inner function and $0 < \alpha < 1$, then $f \in \mathcal{B}_\alpha$ if and only if $f \in \mathcal{D}_{1-\alpha}$.

CHAPTER 4

Radial Limits

Preamble. We study the radial limits and the radial growths of functions in $\mathcal{F}_\alpha$.

Suppose that $0 < \alpha \leq 1$ and f 0 $\mathcal{F}_\alpha$. Theorem 3.5 gives f 0 H^p for $0 < p < \frac{1}{\alpha}$. This implies that f has a radial limit in the direction $e^{i\theta}$ for almost all θ in $[-\pi, \pi]$. The radial limit is denoted by $F(\theta)$, and F 0 $L^p([-\pi, \pi])$ for $0 < p < \frac{1}{\alpha}$.

This result is improved in Theorem 4.2, where we prove that F is weak $L^{1/\alpha}$ on $[-\pi, \pi]$. A similar result is proved for f 0 $\mathcal{F}_0$. The arguments are based on a result of Kolmogoroff, which gives weak L^1 inequalities for analytic functions with range contained in a half-plane. We include a proof of Kolmogoroff's result.

We discuss the nontangential limits of functions in $\mathcal{F}_\alpha$. As a basis for this, we first prove the classical result of Fatou showing that if a bounded analytic function has a radial limit in a particular direction, then it has a nontangential limit in that direction.

An integrability condition based at $e^{i\theta}$ is shown to imply the existence of the radial limit of f 0 $\mathcal{F}_\alpha$ $(\alpha \geq 0)$ in the direction $e^{i\theta}$. The condition is stated in terms of the measure μ representing f and the related total variation measure of arcs on T centered at $e^{i\theta}$.

We discuss subsets of $[-\pi, \pi]$ having zero α-capacity. Such sets play the role of exceptional sets for a number of results about $\mathcal{F}_\alpha$. For example, if $0 \leq \alpha < 1$ and f 0 $\mathcal{F}_\alpha$ then the nontangential limit of f at $e^{i\theta}$ exists for all θ 0 $[-\pi, \pi]$ except possibly for a set having zero α-capacity. Since sets with zero α-capacity are in general "thinner" than sets of Lebesgue measure zero, this improves a fact stated above. When $\alpha > 1$, the situation is different. As will be shown in Chapter 5, if $\alpha > 1$ there is a function f 0 $\mathcal{F}_\alpha$ for which the radial limit in every direction fails to exist.

We study the nontangential limits of $(1-e^{-i\theta}z)^\gamma f(z)$ where $\gamma > 0$ and f 0 $\mathcal{F}_\alpha$ $(\alpha > 0)$. These results are associated with various types of exceptional sets.

Suppose f 0 $\mathcal{F}_\alpha$ and $0 < \alpha \leq 1$. Theorem 3.5 implies that f 0 H^p for $0 < p < 1/\alpha$. Hence

$$F(\theta) \equiv \lim_{r\to 1-} f(re^{i\theta}) \tag{4.1}$$

exists for almost all θ in $[-\pi, \pi]$ and F 0 L^p ($[-\pi, \pi]$) for $0 < p < 1/\alpha$. We call F(θ) the radial limit of f in the direction $e^{i\theta}$, and we set $f(e^{i\theta}) = F(\theta)$. In Chapter 5 we show that when $\alpha > 1$ there exists f 0 $\mathcal{F}_\alpha$ such that for all θ, $f(e^{i\theta})$ does not exist and in fact, $\overline{\lim_{r\to 1-}} |f(re^{i\theta})| = \infty$ for all θ.

We shall obtain weak L^p inequalities for the function F in the case $0 < \alpha \leq 1$. Let m(E) denote the Lebesgue measure of any measurable set E δ $[-\pi, \pi]$. Let $p > 0$. A measurable function F defined on $[-\pi, \pi]$ is called weak L^p provided that there is a positive constant A (depending only on F and p) such that

$$m\left(\{\theta : |F(\theta)| > t\}\right) \leq \frac{A}{t^p} \tag{4.2}$$

for all $t > 0$. If F 0 L^p ($[-\pi, \pi]$) then F is weak L^p, but not conversely. Also, if F is weak L^p and $q < p$, then F 0 L^q ($[-\pi, \pi]$).

We shall show that if $0 < \alpha \leq 1$ and f 0 $\mathcal{F}_\alpha$ then the function F defined by (4.1) is weak $L^{1/\alpha}$. This improves what was stated above. We note that if $0 < \alpha \leq 1$ and $f(z) = \dfrac{1}{(1-z)^\alpha}$, then the boundary function F is weak $L^{1/\alpha}$ but does not belong to $L^{1/\alpha}([-\pi, \pi])$. We also obtain a result of this kind when $\alpha = 0$.

Recall that $\mathcal{P}$ is the set of functions f analytic in $\mathcal{D}$, with $f(0) = 1$ and $\operatorname{Re} f(z) > 0$ for $|z| < 1$.

Lemma 4.1 *Suppose that* f 0 $\mathcal{P}$. *Let* E *be the set of* θ 0 $[-\pi, \pi]$ *for which the limit in* (4.1) *exists. For each* $t > 0$ *let* $E_t = \{\theta$ 0 $E: |F(\theta)| > t\}$. *Then*

$$m(E_t) \leq \frac{4\pi}{t}. \tag{4.3}$$

Proof: Suppose that f 0 $\mathcal{P}$. Since $\mathcal{P}$ δ $\mathcal{F}_1 \subset H^p$ for $0 < p < 1$ the limit in (4.1) exists for almost all θ, that is, $m(E) = 2\pi$. Let $t > 0$ and define the function g by

$$g(z) = 1 + \frac{f(z) - t}{f(z) + t} \qquad (|z| < 1).$$

By assumption $\operatorname{Re} f(z) > 0$ and hence $|g(z) - 1| < 1$ for $|z| < 1$. Let $u = \operatorname{Re} g$. Then $u > 0$. Since u is a bounded harmonic function, $U(\theta) \equiv \lim_{r \to 1-} u(re^{i\theta})$ exists for almost all θ, $U \in L^{\infty}([-\pi, \pi])$ and $u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} U(\theta)\, d\theta$. The function $w \mapsto s$ where $s = 1 + \frac{w - t}{w + t}$ maps $\{w: |w| > t\} \cap \{w: \operatorname{Re} w \geq 0\}$ onto $\{s: |s-1| \leq 1\} \cap \{s: \operatorname{Re} s > 1\}$. Thus $U(\theta) > 1$ for $\theta \in E_t$ and hence

$$u(0) \geq \frac{1}{2\pi} \int_{E_t} U(\theta)\, d\theta \geq \frac{1}{2\pi} m(E_t).$$

Also,

$$u(0) = \operatorname{Re} g(0) = \operatorname{Re} \left\{ 1 + \frac{f(0) - t}{f(0) + t} \right\} = \frac{2}{1 + t}.$$

Therefore $m(E_t) \leq 2\pi\, u(0) = \frac{4\pi}{1+t} \leq \frac{4\pi}{t}$ for $t > 0$.

Theorem 4.2 *Suppose that* $0 < \alpha \leq 1$ and $f \in \mathcal{F}_\alpha$, *and let* F *be defined by* (4.1) *for almost all* θ *in* $[-\pi, \pi]$. *Then* F *is weak* $L^{1/\alpha}$. *Also, there is a positive constant* A *depending only on* α *such that*

$$m(\{\theta : |F(\theta)| > t\}) \leq \frac{A\, \|f\|_{\mathcal{F}_\alpha}^{1/\alpha}}{t^{1/\alpha}} \tag{4.4}$$

for $t > 0$.

Proof: Suppose that $0 < \alpha \leq 1$ and $f(z) = \int_T \frac{1}{(1 - \overline{\zeta} z)^\alpha} d\mu(\zeta)$ for $\mu \in \mathcal{M}$. We may assume that $f \neq 0$ and thus $\|\mu\| > 0$. The Jordan decomposition and (1.10) imply that

$$f = a_1 f_1 - a_2 f_2 + i a_3 f_3 - i a_4 f_4 \tag{4.5}$$

where $a_n \geq 0$, $f_n \in \mathcal{F}_\alpha^*$, and

$$a_1 + a_2 + a_3 + a_4 \leq \sqrt{2}\ \| \mu \|. \tag{4.6}$$

There is a measurable set E δ [−π, π] such that m(E) = 2π and

$$F_n(\theta) \equiv \lim_{r \to 1-} f_n(re^{i\theta})$$

exists for θ 0 E (n = 1, 2, 3, 4). Because $0 < \alpha \leq 1$ it follows as in the proof of Theorem 3.3 that $f_n(\mathbb{D})$ δ $F_\alpha(\mathbb{D})$ where $F_\alpha(z) = \dfrac{1}{(1-z)^\alpha}$. In particular this implies that $f_n(z) \neq 0$ for $|z| < 1$ and hence $g_n = f_n^{1/\alpha}$ is analytic in $\mathbb{D}$. Also $g_n(0) = 1$ and Re $g_n(z) > ½ > 0$ for $|z| < 1$. Thus each function g_n satisfies the assumptions in Lemma 4.1.

Let t > 0, and let E_t = {θ 0 E: |F(θ)| > t}. The relations (4.5) and (4.6) imply that

$$E_t\ \delta \bigcup_{n=1}^{4} \{\theta\ 0\ E : |F_n(\theta)| > t/(\sqrt{2}\,\|\mu\|)\}.$$

Let $G_n(\theta) = \lim_{r \to 1-} g_n(re^{i\theta})$ for θ 0 E. Since

$$\{\ \theta\ 0\ E : |F_n(\theta)| > t/(\sqrt{2}\,\|\mu\|)\ \} = \{\ \theta\ 0\ E : |G_n(\theta)| > \left[t/(\sqrt{2}\,\|\mu\|)\right]^{1/\alpha}\ \}$$

Lemma 4.1 implies that

$$m\big(\{\theta\ 0\ E : |F_n(\theta)| > t/(\sqrt{2}\,\|\mu\|)\}\big) \leq \frac{4\pi(\sqrt{2}\,\|\mu\|)^{1/\alpha}}{t^{1/\alpha}}$$

for n = 1, 2, 3, 4. We conclude that

$$m(E_t) \leq \frac{16\pi\ (\sqrt{2}\,\|\mu\|)^{1/\alpha}}{t^{1/\alpha}}.$$

This proves that F is weak $L^{1/\alpha}$ and it also yields (4.4).

Theorem 4.3 *There is a positive constant* A *such that if* f 0 $\mathcal{F}_0$ *and* $f(0) = 0$ *then*

$$m\left(\{\theta : |F(\theta)| > t\}\right) \le A \exp\left\{\frac{-t}{\sqrt{2}\, \| f \|_{\mathcal{F}_0}}\right\} \tag{4.7}$$

for $t > 0$, *where* F *is defined by* (4.1) *for almost all* θ.

Proof: Assume that f 0 $\mathcal{F}_0$ and $f(0) = 0$. Then

$$f(z) = \int_T \log \frac{1}{1-\overline{\zeta}z}\, d\mu(\zeta) \qquad (|z| < 1)$$

for some μ 0 $\mathcal{M}$. We may assume $f \neq 0$ and thus $\|\mu\| > 0$. Relations (4.5) and (4.6) hold, where

$$f_n(z) = \int_T \log \frac{1}{1-\overline{\zeta}z}\, d\mu_n(\zeta)$$

and μ_n 0 $\mathcal{M}^*$ for n = 1, 2, 3, 4. Since $F_0(z) = \log \dfrac{1}{1-z}$ is a convex univalent function, it follows that $f_n(\mathcal{D})$ δ $F_0(\mathcal{D})$. Also, $f_n(0) = F_0(0) = 0$. Hence f_n is subordinate to F_0 and therefore $f_n = F_0 \circ \varphi_n$ where φ_n is analytic in $\mathcal{D}$, $\varphi_n(0) = 0$ and $|\varphi_n(z)| < 1$ for $|z| < 1$. Let $p_n = \dfrac{1}{1-\varphi_n}$ for n = 1, 2, 3, 4. Then Re $p_n(z) > ½$ for $|z| < 1$, $p_n(0) = 1$ and $f_n(z) = \log p_n(z)$ for $|z| < 1$. There is a measurable set $E \subset [-\pi, \pi]$ such that $m(E) = 2\pi$ and if θ 0 E then $\Phi_n(\theta) \equiv \lim_{r\to 1-} \varphi_n(re^{i\theta})$ exists and $\Phi_n(\theta) \neq 1$ for n = 1, 2, 3, 4.

Let $t > 0$ and let $E_t = \{\theta$ 0 E: $|F(\theta)| > t\}$. As in the proof of Theorem 4.2,

$$E_t \;\delta\; \bigcup_{n=1}^{4} \{\theta \;\, 0\; E: |F_n(\theta)| > \frac{t}{\sqrt{2}\,\|\mu\|}\}.$$

Let $P_n(\theta) = \lim_{r\to 1-} p_n(re^{i\theta})$ for θ 0 E and n = 1, 2, 3, 4. If θ 0 E and $|F_n(\theta)| > \dfrac{t}{\sqrt{2}\,\|\mu\|}$ for some n, then $|\log P_n(\theta)| > \dfrac{t}{\sqrt{2}\,\|\mu\|}$. For θ 0 E, Re $P_n(\theta) \geq ½ > 0$ and thus

$$|\log P_n(\theta)| \le |\log|P_n(\theta)|| + \pi/2$$

for such θ. Therefore if θ 0 E and $t \geq t_0 \equiv \sqrt{2}\, \|\mu\| \left(\log 2 + \frac{\pi}{2} \right)$, then

$$| P_n(\theta) | > e^{-\pi/2} \exp \left\{ \frac{t}{\sqrt{2}\, \|\mu\|} \right\}.$$

Lemma 4.1 implies that

$$m\left(\left\{ \theta \cap E : | P_n(\theta) | > e^{-\pi/2} \exp \left[t / (\sqrt{2}\, \|\mu\|) \right] \right\} \right) \leq \frac{4\pi e^{\pi/2}}{\exp \left[\frac{t}{\sqrt{2}\, \|\mu\|} \right]}$$

for $t \geq t_0$. Since this holds for n = 1, 2, 3, 4 we conclude that

$m(E_t) \leq A \exp \left\{ \frac{-t}{\sqrt{2}\, \|\mu\|} \right\}$ for $t \geq t_0$, where $A = 16\pi e^{\pi/2}$. If $0 < t < t_0$, then

$$m(E_t) \leq 2\pi = \frac{A}{4} \exp \left\{ \frac{-t_0}{\sqrt{2}\, \|\mu\|} \right\} \leq \frac{A}{4} \exp \left[\frac{-t}{\sqrt{2}\, \|\mu\|} \right].$$

Therefore $m(E_t) \leq A \exp \left\{ \frac{-t}{\sqrt{2}\, \|\mu\|} \right\}$ for all $t > 0$. This implies (4.7).

Next we study the radial and the nontangential limits of functions in $\mathcal{F}_\alpha$. For $-\pi \leq \theta \leq \pi$ and $0 < \gamma < \pi$ let $S(\theta, \gamma) = \{z: 0 < | z - e^{i\theta}| \leq \cos \frac{\gamma}{2}\} \cap A(\theta, \gamma)$, where $A(\theta, \gamma)$ denotes the closed angular region that has vertex $e^{i\theta}$, includes 0, has an opening γ and is symmetric about the line through 0 and $e^{i\theta}$. We call $S(\theta, \gamma)$ the Stolz angle with vertex $e^{i\theta}$ and opening γ. A function f: $\mathcal{D} \to \mathbb{C}$ is said to have a nontangential limit at $e^{i\theta}$ provided that

$$\lim_{\substack{z \to e^{i\theta} \\ z \in S(\theta, \gamma)}} f(z)$$

exists for every γ ($0 < \gamma < \pi$).

The next two lemmas show that for functions in certain families, the existence of a radial limit implies the existence of a nontangential limit.

Lemma 4.4 *Suppose that* f 0 H^∞ *and* $\lim_{r\to 1-} f(re^{i\theta})$ *exists for some* θ *in* $[-\pi, \pi]$. *Then* f *has a nontangential limit at* $e^{i\theta}$.

Proof: We may assume that $\theta = 0$. Let $L = \lim_{r\to 1-} f(r)$ where f 0 H^∞. For $n = 1, 2, \ldots$ let $f_n(z) = f(w)$ where $w = (1 - \frac{1}{n}) + \frac{z}{n}$. Then f_n 0 H^∞ and $\| f_n \|_{H^\infty} \le \| f \|_{H^\infty}$ for $n = 1, 2, \ldots$.

We claim that $f_n \to L$ uniformly on compact subsets of $\mathbb{D}$. On the contrary, suppose that this is not true. Then there exist numbers r and ε, a sequence of integers $\{n_k\}$ and a sequence of complex numbers $\{z_k\}$ such that $0 < r < 1$, $\varepsilon > 0$, $n_k \to \infty$ as $k \to \infty$, $| z_k | \le r$ and $| f_{n_k}(z_k) - L | \ge \varepsilon$ for $k = 1,2,\ldots$. Since $\{ f_{n_k} \}$ is locally bounded, Montel's theorem implies that there is a subsequence of $\{ f_{n_k} \}$ which converges uniformly on compact subsets of $\mathbb{D}$ and the limit function g is analytic in $\mathbb{D}$. The definition of f_n and the fact that f has the radial limit L implies that $\lim_{n\to\infty} f_n(z) = L$ for every z in (0,1). Therefore the identity theorem implies that $g(z) = L$ for $|z| < 1$. The uniform convergence of the subsequence of $\{ f_{n_k} \}$ to L on $\{z: |z| \le r\}$ contradicts the assertion that $| f_{n_k}(z_k) - L | \ge \varepsilon$ for $k = 1, 2, \ldots$. This proves our claim.

Let $0 < \gamma < \pi$ and let $S = S(0, \gamma)$. Let

$$V = S \cap \{z: \tfrac{1}{2} \cos(\gamma/2) \le | z - 1 | \le \cos(\gamma/2)\}.$$

Then V is a compact subset of $\mathbb{D}$ and hence $f_n \to L$ uniformly on V. Assume that $\varepsilon > 0$. There is an integer N such that $| f_n(z) - L | < \varepsilon$ for $n \ge N$ and z 0 V. Then

$$\{w : w = (1 - \tfrac{1}{n}) + \tfrac{z}{n},\ n \ge N,\ z \ 0\ V\} = \{w : 0 < | w - 1 | \le \tfrac{1}{N} \cos(\tfrac{\gamma}{2})\} \cap S.$$

Let $\delta = \cos(\frac{\gamma}{2}) / N$. If w 0 S and $0 < |w-1| \le \delta$, then $f(w) = f_n(z)$ for some z 0 V and $n \ge N$. Thus $| f(w) - L| < \varepsilon$. This proves that $\lim_{\substack{w\to 1 \\ w\in S}} f(w) = L$.

Lemma 4.5 *Suppose that* $0 \le \alpha \le 1$, f 0 F_α^* *and* $\lim_{r\to 1-} f(re^{i\theta})$ *exists for some* θ *in* $[-\pi, \pi]$. *Then* f *has a nontangential limit at* $e^{i\theta}$.

Proof: Suppose that $0 < \alpha \leq 1$ and $f \in \mathcal{F}_\alpha^*$. Then f is subordinate to F_α where

$$F_\alpha(z) = \frac{1}{(1-z)^\alpha} \qquad (|z| < 1).$$

Hence

$$f(z) = F_\alpha(\varphi(z)) \qquad (|z| < 1) \tag{4.8}$$

where φ is analytic in $\mathcal{D}$, $|\varphi(z)| < 1$ for $|z| < 1$ and $\varphi(0) = 0$. Since $f(\mathcal{D}) \subseteq F_\alpha(\mathcal{D})$ and $F_\alpha(z) \neq 0$, we have $f(z) \neq 0$. Hence (4.8) yields

$$\varphi(z) = 1 - [f(z)]^{-1/\alpha}. \tag{4.9}$$

Because F_α is analytic and univalent in $\mathcal{D}$, $F_\alpha(\mathcal{D})$ is convex and $F_\alpha(z)$ is real for all real $z \in \mathcal{D}$, it follows that

$$\inf_{|z|<1} \operatorname{Re} F_\alpha(z) = \inf_{-1<x<1} \operatorname{Re} F_\alpha(x).$$

Thus $\operatorname{Re} F_\alpha(z) > (\tfrac{1}{2})^\alpha$ for $|z| < 1$. Assume that $L = \lim_{r\to 1-} f(re^{i\theta})$ exists for some θ in $[-\pi, \pi]$. Because $\operatorname{Re} F_\alpha(z) > (\tfrac{1}{2})^\alpha$, the relation (4.8) implies that $\operatorname{Re} f(z) > (\tfrac{1}{2})^\alpha$ for $|z| < 1$. Hence $\operatorname{Re} L \geq (\tfrac{1}{2})^\alpha > 0$. Therefore (4.9) implies $\lim_{r\to 1-} \varphi(re^{i\theta}) = 1 - L^{-1/\alpha}$. Let $M = 1 - L^{-1/\alpha}$. Since φ has a radial limit at $e^{i\theta}$, Lemma 4.4 implies that φ has a nontangential limit at $e^{i\theta}$. Since $M \neq 1$ and F_α is continuous in $\overline{D} \setminus \{1\}$, (4.8) shows that f also has a nontangential limit at $e^{i\theta}$. This proves the lemma when $0 < \alpha \leq 1$.

The case $\alpha = 0$ can be treated in a similar way. If $F_0(z) = \log \dfrac{1}{1-z}$ then (4.8) is replaced by $f(z) = f(0) + F_0(\varphi(z))$ and (4.9) is replaced by $\varphi(z) = 1 - \exp(f(0)) \exp(-f(z))$. □

Let $\mu \in \mathcal{M}$ and suppose that μ is real-valued and nonnegative. As in Chapter 1, μ associates with a real-valued function g defined on $[-\pi, \pi]$ or on any interval of the form $[\theta - \pi, \theta + \pi]$. The function g is nondecreasing and continuous from the right. It follows that if

$$f(z) = \int_T \frac{1}{(1-\overline{\zeta} z)^\alpha} \, d\mu(\zeta) \qquad (|z| < 1) \tag{4.10}$$

where $\alpha > 0$, then f can be expressed as a Lebesgue-Stieltjes integral

$$f(z) = \int_{\theta-\pi}^{\theta+\pi} \frac{1}{(1 - e^{-it}z)^{\alpha}} \, dg(t) \qquad (|z| < 1). \tag{4.11}$$

In general, if $\alpha > 0$ and $f \; 0 \; \mathcal{F}_\alpha$ then (4.10) holds for some $\mu \; 0 \; \mathcal{M}$. The measure μ associates with a complex-valued function g of bounded variation on $[\theta - \pi, \theta + \pi]$, where $g(\theta - \pi) = 0$ and (4.11) holds. We generally assume that g is extended to $(-\infty, \infty)$ by letting

$$g(t + \theta + \pi) = g(t + \theta - \pi) + g(\theta + \pi).$$

For $|\zeta| = 1$ and $0 < x \leq \pi$ let $I(\zeta, x)$ denote the closed arc on T centered at ζ and having length 2x. The function ω is defined on $[0, \pi]$ by

$$\omega(x) = \omega(x, \zeta; \mu) = |\mu|\big(I(\zeta, x)\big) \tag{4.12}$$

for $0 < x \leq \pi$ and $\omega(0) = |\mu|(\{\zeta\})$. If μ is nonnegative, $0 < x \leq \pi$ and $\zeta = e^{i\theta}$, then

$$\omega(x) = \omega(x, e^{i\theta}; \mu) = \mu(I(\zeta, x)) \geq g(\theta + x) - g(\theta - x) \geq g(\theta + x) - g(\theta).$$

The behavior of $f \; 0 \; \mathcal{F}_\alpha$ for z near $e^{i\theta}$ depends upon the behavior of ω near 0. One example of this is the next result about radial limits.

Theorem 4.6 *Suppose that* $\alpha \geq 0$, $f \; 0 \; \mathcal{F}_\alpha$ *and* f *is represented in* $\mathcal{F}_\alpha$ *by* $\mu \; 0 \; \mathcal{M}$. *Let* ω *be defined by* (4.12) *where* $\zeta = e^{i\theta}$. *If*

$$\int_0^{\pi} \frac{\omega(t)}{t^{\alpha+1}} \, dt < \infty \tag{4.13}$$

then $\lim_{r \to 1-} f(re^{i\theta})$ *exists. If* $0 \leq \alpha \leq 1$ *then* (4.13) *implies that* f *has a nontangential limit at* $e^{i\theta}$.

Proof: Suppose that $\alpha > 0$, $f \; 0 \; \mathcal{F}_\alpha$ and f is represented by μ. The Jordan decomposition theorem and (1.10) give $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ where $\mu_n \geq 0$ for $n = 1, 2, 3, 4$ and $\sum_{n=1}^{4} \mu_n(E) \leq \sqrt{2}\,|\mu|(E)$ for every set E which is

μ-measurable. Each measure μ_n is associated with a real-valued nondecreasing function g_n on $[\theta - \pi, \theta + \pi]$ as described above. Let $\omega_n(t) = \omega(t, e^{i\theta}; \mu_n)$. If g_n does not have a (jump) discontinuity at $\theta - t$ then

$$\omega_n(t) = g_n(\theta + t) - g_n(\theta - t) = |\, g_n(\theta + t) - g_n(\theta)\,| + |\, g_n(\theta) - g_n(\theta - t)\,|.$$

Since g_n is continuous except possibly on a countable set, this equality holds for almost all t. Hence

$$\begin{aligned} &\int_{-\pi}^{\pi} \frac{|\,g_n(\theta + t) - g_n(\theta)\,|}{|\,t\,|^{\alpha+1}}\, dt \\ &= \int_0^{\pi} \frac{|\,g_n(\theta + t) - g_n(\theta)\,|}{t^{\alpha+1}}\, dt + \int_0^{\pi} \frac{|\,g_n(\theta) - g_n(\theta - t)\,|}{t^{\alpha+1}}\, dt \\ &= \int_0^{\pi} \frac{\omega_n(t)}{t^{\alpha+1}}\, dt. \end{aligned}$$

Let $g = g_1 - g_2 + ig_3 - ig_4$. Then g associates with the measure μ and

$$\int_{-\pi}^{\pi} \frac{|\,g(\theta + t) - g(\theta)\,|}{|\,t\,|^{\alpha+1}}\, dt \le \int_0^{\pi} \left\{ \frac{1}{t^{\alpha+1}} \sum_{n=1}^{4} \omega_n(t) \right\} dt \le \int_0^{\pi} \frac{\sqrt{2}\ \omega(t)}{t^{\alpha+1}}\, dt.$$

Hence the assumption (4.13) implies that

$$\int_{-\pi}^{\pi} \frac{|\,g(\theta + t) - g(\theta)\,|}{|\,t\,|^{\alpha+1}}\, dt < \infty. \tag{4.14}$$

From (4.11) we have

$$f(z) = \int_{\theta-\pi}^{\theta+\pi} \frac{1}{(1 - e^{-it} z)^{\alpha}}\, d\big(g(t) - g(\theta)\big).$$

Integration by parts yields

$$f(z) = \frac{g(\theta + \pi)}{(1 + e^{-i\theta} z)^{\alpha}} + i\alpha \int_{\theta-\pi}^{\theta+\pi} K(e^{-it} z)\, [g(t) - g(\theta)]\, dt \tag{4.15}$$

where

$$K(z) = \frac{z}{(1-z)^{\alpha+1}}. \tag{4.16}$$

Hence

$$f(z) = \frac{g(\theta+\pi)}{(1+e^{-i\theta}z)^{\alpha}} + i\alpha \int_{-\pi}^{\pi} K(e^{-i(\theta+t)}z)\,[g(\theta+t) - g(\theta)]\,dt. \tag{4.17}$$

Therefore it suffices to show that $\lim_{r\to 1-} F(r)$ exists where

$$F(z) = \int_{-\pi}^{\pi} K(e^{-it}z)\,[g(\theta+t) - g(\theta)]\,dt. \tag{4.18}$$

If $\frac{1}{2} \le r < 1$ and $-\pi \le t \le \pi$ then $|1 - re^{it}|^2 \ge (1-r)^2 + \frac{2}{\pi^2}\,t^2 \ge \frac{2}{\pi^2}\,t^2$. Hence, if $\frac{1}{2} \le r < 1$ then $|K(re^{-it})| \le (\frac{\pi}{\sqrt{2}})^{\alpha+1}\,\frac{1}{|t|^{\alpha+1}}$ for $0 < |t| < \pi$. Because of (4.14) this shows that the integrand in (4.18) is dominated by an integrable function for $\frac{1}{2} \le r < 1$. For $0 < |t| < \pi$, $\lim_{r\to 1-} K(re^{-it}) = \frac{e^{-it}}{(1-e^{-it})^{\alpha+1}}$. Also (4.14) implies

$$\int_{-\pi}^{\pi} \left|\frac{e^{-it}}{(1-e^{-it})^{\alpha+1}}\right| [g(\theta+t) - g(\theta)]\,dt \le \int_{-\pi}^{\pi} \frac{|g(\theta+t) - g(\theta)|}{|2\sin(t/2)|^{\alpha+1}}\,dt < \infty.$$

Therefore the Lebesgue dominated convergence theorem implies that

$$\lim_{r\to 1-} F(r) = \int_{-\pi}^{\pi} \lim_{r\to 1-} K(re^{-it})\,[g(\theta+t) - g(\theta)]\,dt$$

exists. This completes the proof about the radial limit when $\alpha > 0$.

The same argument applies when $\alpha = 0$, starting with

$$f(z) = f(0) + \int_{\theta-\pi}^{\theta+\pi} \log \frac{1}{(1-e^{-it}z)}\, dg(t).$$

To prove the last assertion in the theorem, let f 0 $\mathcal{F}_\alpha$ where $0 \le \alpha \le 1$. Then $f = a_1 f_1 - a_2 f_2 - ia_3 f_3 - ia_4 f_4$ where $a_n \ge 0$ and f_n 0 $\mathcal{F}_\alpha^*$ (n = 1, 2, 3, 4). By the previous argument, (4.13) implies that $\lim_{r\to 1-} f_n(re^{i\theta})$ exists for n = 1, 2, 3 and 4. Since $0 \le \alpha \le 1$ Lemma 4.5 implies that f_n has a nontangential limit at $e^{i\theta}$ for n = 1, 2, 3 and 4. Hence f has a nontangential limit at $e^{i\theta}$.

For each α, where $0 \le \alpha < 1$, the function P_α is defined as follows. Assume that θ is not an integral multiple of 2π. If $0 < \alpha < 1$ let

$$P_\alpha(\theta) = \frac{1}{|\sin(\theta/2)|^\alpha} \tag{4.19}$$

and if $\alpha = 0$ let

$$P_0(\theta) = \log \frac{1}{|\sin(\theta/2)|}. \tag{4.20}$$

Note that $P_\alpha(\theta) \ge 0$ and $\int_{-\pi}^{\pi} P_\alpha(\theta)\, d\theta < \infty$ for every α. Let E be a nonempty Borel subset of [–π, π]. The set E is said to have positive α-capacity provided that there exists a probability measure μ supported on E such that

$$\sup_\theta \int_{-\pi}^{\pi} P_\alpha(\theta - t)\, d\mu(t) < \infty. \tag{4.21}$$

If there is no such measure for which (4.21) holds, then E is said to have zero α-capacity. Intuitively, E has positive α-capacity if E has sufficient "thickness" so that a measure can be distributed over E in such a way as to "cancel" the singularities generated by P_α. We use the notations $C_\alpha(E) > 0$ to mean that E has positive α-capacity and $C_\alpha(E) = 0$ to mean that E has zero α-capacity. There is a more general meaning for $C_\alpha(E)$, the α-capacity of a Borel set E, but it is not used here.

The basic properties of zero α-capacity are given in the next proposition.

Proposition 4.7 *Let* E *and* F *be Borel subsets of* $[-\pi, \pi]$ *and let* $0 \leq \alpha < 1$ *and* $0 \leq \beta < 1$.

(a) *If* $E \delta F$ *and* $C_\alpha(F) = 0$, *then* $C_\alpha(E) = 0$.

(b) *If* $C_\alpha(E) = 0$ *and* $\beta > \alpha$, *then* $C_\beta(E) = 0$.

(c) *If* $C_\alpha(E) = 0$ *and* $C_\alpha(F) = 0$, *then* $C_\alpha(E \cup F) = 0$.

(d) *If* E *is a finite set or a countably infinite set, then* $C_\alpha(E) = 0$ *for all* α.

(e) *If* $C_\alpha(E) = 0$ *for some* α, *then the Lebesgue measure of* E *is zero.*

Proof: Suppose that $C_\alpha(E) > 0$ and $E \delta F$. Then there is a probability measure μ supported on E such that (4.21) holds. Since $E \delta F$, μ induces a probability measure υ on F defined by $\upsilon(B) = \mu(B \cap E)$ for each Borel set $B \delta F$. Then

$$\int_F P_\alpha(\theta - t)\, d\upsilon(t) = \int_E P_\alpha(\theta - t)\, d\mu(t)$$

for all θ. Hence

$$\sup_\theta \int_E P_\alpha(\theta - t)\, d\upsilon(t) = \sup_\theta \int_{-\pi}^{\pi} P_\alpha(\theta - t)\, d\upsilon(t) = \sup_\theta \int_{-\pi}^{\pi} P_\alpha(\theta - t)\, d\mu(t) < \infty$$

and thus $C_\alpha(F) > 0$. This proves (a).

Suppose that $0 < \beta < 1$ and $C_\beta(E) > 0$. Then there is a probability measure μ supported on E such that $\sup_\theta \int_E P_\beta(\theta - t)\, d\mu(t) < \infty$. Suppose that $0 < \alpha < \beta$. Then $P_\alpha(\theta) \leq P_\beta(\theta)$. Therefore $\sup_\theta \int_E P_\alpha(\theta - t)\, d\mu(t) < \infty$ and hence $C_\alpha(E) > 0$. This proves (b) when $0 < \alpha < \beta < 1$. The inequality $y^\beta \geq 1 + \beta \log y$ holds for $y \geq 1$ and $0 < \beta < 1$. Hence $P_0(\theta) \leq \frac{1}{\beta} P_\beta(\theta)$. This implies that if $C_\beta(E) > 0$ for some β $(0 < \beta < 1)$ then $C_0(E) > 0$.

Let $G = E \cup F$ and suppose that $C_\alpha(G) > 0$. Then there is a probability measure μ supported on G such that $\sup_\theta \int_G P_\alpha(\theta - t)\, d\mu(t) < \infty$. Since $1 = \int_G d\mu(t) \le \int_E d\mu(t) + \int_F d\mu(t)$, we may assume that $\int_E d\mu(t) \ge \tfrac{1}{2} > 0$. Let υ be the measure defined by $\upsilon(H) = \mu(E \cap H)$ for each Borel set H and let $\lambda = (1/b)\upsilon$ where $b = \int_E d\mu(t)$. Then λ is a probability measure supported on E and

$$\int_E P_\alpha(\theta - t)\, d\lambda(t) = \frac{1}{b}\int_E P_\alpha(\theta - t)\, d\upsilon(t)$$
$$\le 2\int_E P_\alpha(\theta - t)\, d\upsilon(t)$$
$$\le 2\int_G P_\alpha(\theta - t)\, d\upsilon(t).$$

Therefore $\sup_\theta \int_E P_\alpha(\theta - t)\, d\lambda(t) < \infty$ and hence $C_\alpha(E) > 0$. This proves (c).

Clearly $C_\alpha(E) = 0$ when E consists of a single point. Hence (c) implies that $C_\alpha(E) = 0$ for any finite set E. Suppose that E is countably infinite and let $E = \bigcup_{n=1}^{\infty} \{t_n\}$ where $t_n \ne t_m$ for $n \ne m$. Let μ be a probability measure supported on E. Then there exists k such that $\mu(\{t_k\}) > 0$. Therefore

$$\int_E P_\alpha(\theta - t)\, d\mu(t) = \sum_{n=1}^{\infty} P_\alpha(\theta - t_n)\, \mu(\{t_n\}) \ge P_\alpha(\theta - t_k)\, \mu(\{t_k\}).$$

Since $\mu(\{t_k\}) > 0$ and $\lim_{\theta \to t_k} P_\alpha(\theta - t_k) = \infty$, this implies that

$$\sup_\theta \int_E P_\alpha(\theta - t)\, d\mu(t) = \infty.$$

Hence $C_\alpha(E) = 0$. This proves (d).

Finally suppose that $m(E) > 0$. Let g denote the characteristic function of E and let μ be defined by $d\mu(t) = \frac{1}{m(E)} g(t)\, dt$. Then μ is a probability measure supported on E. Using the periodicity of P_α we obtain

$$\int_E P_\alpha(\theta - t)\, d\mu(t) = \frac{1}{m(E)} \int_E P_\alpha(\theta - t)\, dt$$

$$\leq \frac{1}{m(E)} \int_{-\pi}^{\pi} P_\alpha(\theta - t)\, dt$$

$$= \frac{1}{m(E)} \int_{-\pi}^{\pi} P_\alpha(t)\, dt < \infty.$$

Therefore $C_\alpha(E) > 0$, and this proves (e).

Suppose that Q(θ) is a proposition for each θ in [–π, π]. We say that Q(θ) holds α quasi-everywhere provided that there is a Borel set E δ [–π, π] such that C_α([–π, π] (E) = 0 and Q(θ) holds for all θ 0 E.

The function g described in the next lemma is defined on [–π, π]. As usual we assume that g is extended to (–∞, ∞) by letting $g(t + \pi) = g(t - \pi) + g(\pi)$ for all real t.

Lemma 4.8 *Let g be a real-valued nondecreasing function on* [–π, π] *and let* $0 \leq \alpha < 1$. *Then*

$$\int_0^{\pi} \frac{g(\theta + t) - g(\theta - t)}{t^{\alpha+1}}\, dt < \infty \tag{4.22}$$

α *quasi-everywhere.*

Proof: For $-\pi \leq \theta \leq \pi$ let

$$I_\alpha(\theta) = \int_{-\pi}^{\pi} P_\alpha(\theta - t)\, dg(t). \tag{4.23}$$

Let E = {θ: $I_\alpha(\theta) = \infty$}. Then E is a Borel subset of [–π, π]. We claim that $C_\alpha(E) = 0$. On the contrary, suppose that $C_\alpha(E) > 0$. Then there is a probability measure μ supported on E such that (4.21) holds. Hence Fubini's theorem yields

$$\int_E I_\alpha(\theta)\ d\mu(\theta) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P_\alpha(\theta - t)\ dg(t)\ d\mu(\theta)$$

$$= \int_{-\pi}^{\pi}\left\{\int_{-\pi}^{\pi} P_\alpha(\theta - t)\ d\mu(\theta)\right\} dg(t)$$

$$= \int_{-\pi}^{\pi}\left\{\int_{-\pi}^{\pi} P_\alpha(t - \theta)\ d\mu(\theta)\right\} dg(t)$$

$$\leq \left\{\sup_t \int_{-\pi}^{\pi} P_\alpha(t - \theta)\ d\mu(\theta)\right\} \int_{-\pi}^{\pi} dg(t) < \infty.$$

Since μ is supported on E and $I_\alpha(\theta) = \infty$ for all θ 0 E, $\int_E I_\alpha(\theta)\ d\mu(\theta) = \infty$. This contradicts the previous inequality. Hence $C_\alpha(E) = 0$.

Let F = $[-\pi, \pi]$ (E. Suppose that θ 0 F. Then $I_\alpha(\theta) < \infty$ and

$$\int_{-\pi}^{\pi} P_\alpha(t)\ dg(\theta + t) = \int_{-\pi}^{\pi} P_\alpha(\theta - t)\ dg(t) < \infty.$$

Let $\varepsilon > 0$. The last inequality implies that there is a positive real number δ such that $\int_{-2\delta}^{2\delta} P_\alpha(t)\ dg(\theta + t) < \varepsilon$. The monotonicity of P_α and of g implies that

$$\int_{-2\delta}^{2\delta} P_\alpha(t)\ dg(\theta + t) \geq P_\alpha(2\delta) \int_{-2\delta}^{2\delta} dg(\theta + t)$$

$$\geq P_\alpha(2\delta)\left[g(\theta + \delta) - g(\theta - \delta)\right]$$

$$= P_\alpha(2\delta)\ h(\theta, \delta)$$

where, in general, $h(\theta,t) = g(\theta + t) - g(\theta - t)$. We have shown that $h(\theta, \delta) < \varepsilon$ ∍ $P_\alpha(2\delta)$. Because of the local behavior of P_α near 0 and because $g(\theta + \delta) - g(\theta) \leq g(\theta + \delta) - g(\theta - \delta)$, we conclude that

$$\lim_{t\to 0} \frac{g(\theta+t) - g(\theta)}{|\,t\,|^{\alpha}} = 0. \tag{4.24}$$

Integration by parts yields

$$\int_{\delta}^{\pi} P_\alpha(t)\; dg(\theta+t) = g(\theta+\pi)P_\alpha(\pi) - g(\theta+\delta)P_\alpha(\delta) - \int_{\delta}^{\pi} g(\theta+t)P'_\alpha(t)\; dt$$

and

$$\int_{-\pi}^{-\delta} P_\alpha(t)\; dg(\theta+t) = g(\theta-\delta)P_\alpha(-\delta) - g(\theta-\pi)P_\alpha(-\pi) - \int_{-\pi}^{-\delta} g(\theta+t)P'_\alpha(t)\; dt.$$

Hence $\int_{-\pi}^{\pi} P_\alpha(t)\; dg(\theta+t) = \int_{-\delta}^{\delta} P_\alpha(t)\; dg(\theta+t) + [g(\theta+\pi) - g(\theta-\pi)]\; P_\alpha(\pi)$

$$+ [g(\theta-\delta) - g(\theta+\delta)]\; P_\alpha(\delta) - \int_{\delta}^{\pi} [g(\theta+t) - g(\theta-t)]\; P'_\alpha(t)\; dt.$$

Because $\int_{-\pi}^{\pi} P_\alpha(t)\; dg(\theta+t) < \infty$, (4.24) implies that we may let $\delta \to 0$ in the previous equality. Therefore

$$\lim_{\delta\to 0+} \int_{\delta}^{\pi} h(\theta,t)\; [-P'_\alpha(t)]\; dt < \infty. \tag{4.25}$$

If $\alpha > 0$ then $-P'_\alpha(t) = \dfrac{\alpha}{2}\, \dfrac{\cos(t/2)}{[\sin(t/2)]^{\alpha+1}}$ for $t > 0$, and $-P'_0(t) = \frac{1}{2}\cot(t/2)$ for $t > 0$. Thus (4.25) yields

$$\lim_{\delta\to 0+} \int_{\delta}^{\pi} \frac{h(\theta,t)}{t^{\alpha+1}}\; dt < \infty \tag{4.26}$$

for $0 \le \alpha < 1$. Since $h(\theta, t) / t^{\alpha+1} \ge 0$, (4.26) implies (4.22).

We have shown that if θ 0 F then (4.22) holds. Also E = [$-\pi$, π] (F and $C_\alpha(E) = 0$.

As noted in Chapter 3, if f 0 $\mathcal{F}_\alpha$ for some α where $0 \le \alpha \le 1$, then f belongs to H^p for suitable values of p. Consequently f has a nontangential limit at $e^{i\theta}$ for almost all θ in [$-\pi$, π]. The next theorem gives an improvement of this result when $0 \le \alpha < 1$. Exceptional sets having measure zero are replaced by exceptional sets having α-capacity zero.

Theorem 4.9 *Suppose that* $0 \le \alpha < 1$ *and* f 0 $\mathcal{F}_\alpha$. *Then f has a nontangential limit at* $e^{i\theta}$ α *quasi-everywhere for* θ 0 [$-\pi$, π].

Proof: Suppose that $0 < \alpha < 1$ and f 0 $\mathcal{F}_\alpha$. There exist nonnegative real numbers a_n and measures μ_n 0 $\mathcal{M}^*$ for n = 1, 2, 3, 4 such that

$$f = a_1 f_1 - a_2 f_2 + ia_3 f_3 - ia_4 f_4$$

where

$$f_n(z) = \int_T \frac{1}{(1-\overline{\zeta}z)^\alpha}\, d\mu_n(\zeta) \qquad (|z|<1). \tag{4.27}$$

Let g_n denote the real-valued nondecreasing function on [$-\pi$, π] which we have associated with μ_n. Then

$$f_n(z) = \int_{-\pi}^{\pi} \frac{1}{(1-e^{-it}z)^\alpha}\, dg_n(t) \qquad (|z|<1). \tag{4.28}$$

By Lemma 4.8, for each n there is a Borel set F_n δ [$-\pi$, π] such that

$$\int_0^{\pi} \frac{g_n(\theta+t) - g_n(\theta-t)}{t^{\alpha+1}}\, dt < \infty \tag{4.29}$$

for all θ 0 F_n and $C_\alpha(E_n) = 0$ where E_n = [$-\pi$, π] (F_n. Since g_n is nondecreasing, (4.29) implies that

$$\int_{-\pi}^{\pi} \frac{|g_n(\theta+t) - g_n(\theta-t)|}{|t|^{\alpha+1}}\, dt < \infty. \tag{4.30}$$

As shown in the proof of Theorem 4.6, $\int_0^{\pi} \frac{\omega_n(t)}{t^{\alpha+1}}\, dt$ equals the integral in (4.30).

Hence Theorem 4.6 implies that $\lim_{r\to 1-} f_n(re^{i\theta})$ exists. Because of Lemma 4.5 this shows that f_n has a nontangential limit at $e^{i\theta}$ for all θ 0 F_n.

Let $F = \bigcap_{n=1}^{4} F_n$ and let E = [–π, π] (F. Then $E = \bigcup_{n=1}^{4} E_n$, and since $C_\alpha(E_n) = 0$ for n = 1, 2, 3, 4, Proposition 4.7 (part c) implies that $C_\alpha(E) = 0$. If θ 0 F then f_n has a nontangential limit at $e^{i\theta}$ for n = 1, 2, 3, 4. Therefore *f* has a nontangential limit at $e^{i\theta}$ for every θ 0 F. This proves the theorem when $0 < \alpha < 1$. The argument in the case α = 0 is the same. □

Next we examine the radial and nontangential growth of functions in $\mathcal{F}_\alpha$. We find that certain growths can be associated with certain exceptional sets. If α > 0 and *f* 0 $\mathcal{F}_\alpha$ then $|f(z)| \le \frac{\|f\|_{\mathcal{F}_\alpha}}{(1-|z|)^\alpha}$ and hence $|f(z)| = O\left(\frac{1}{(1-r)^\alpha}\right)$ where $|z| = r$. If $f(z) = \frac{1}{(1-z)^\alpha}$, then this maximal growth occurs only in the direction θ = 0. The next result shows that, in general, this maximal growth is permissible for at most a countable set of radial directions.

Theorem 4.10 *If* α > 0 *and* *f* 0 $\mathcal{F}_\alpha$ *then* $(1 - e^{-i\theta}z)^\alpha f(z)$ *has the nontangential limit zero at* $e^{i\theta}$ *for all* θ *in* [–π, π] *except possibly for a finite or countably infinite set. Conversely, let* α > 0 *and let* E *be a finite or countably infinite subset of* [–π, π]. *Then there is a function* *f* 0 $\mathcal{F}_\alpha$ *such that* $(1 - e^{-i\theta}z)^\alpha f(z)$ *has a nontangential limit for all* θ, *and this limit is zero if and only if* θ ⌠ E.

Proof: Suppose that α > 0, *f* 0 $\mathcal{F}_\alpha$, and *f* is nonconstant. Then

$$f(z) = \int_T \frac{1}{(1-\overline{\zeta}z)^\alpha}\, d\mu(\zeta) \qquad (|z| < 1)$$

for some μ 0 $\mathcal{M}$. Let $0 < \gamma < \pi$ and let S = S(θ, γ). There is a positive constant A such that $|z - e^{i\theta}| \le A(1 - |z|)$ for z 0 S. Hence, if z 0 S and |ζ| = 1, then $\left|\frac{(1-e^{-i\theta}z)^\alpha}{(1-\overline{\zeta}z)^\alpha}\right| \le A^\alpha$. Also, if |ζ| = 1 and $\zeta \ne e^{i\theta}$ then

$$\lim_{\substack{z\to e^{i\theta}\\ |z|<1}} \left(\frac{1-e^{-i\theta}z}{1-\overline{\zeta}z}\right)^\alpha = 0, \text{ and if } \zeta = e^{i\theta} \text{ then } \left(\frac{1-e^{-i\theta}z}{1-\overline{\zeta}z}\right)^\alpha = 1. \text{ Hence}$$

$$\lim_{\substack{z\to e^{i\theta}\\ z\in S}} \left[(1 - e^{-i\theta}z)^{\alpha} f(z)\right] = \lim_{\substack{z\to e^{i\theta}\\ z\in S}} \int_T \left(\frac{1-e^{-i\theta}z}{1-\bar{\zeta}z}\right)^{\alpha} d\mu(\zeta) = \mu\Big(\{e^{i\theta}\}\Big),$$

because the integrand is bounded. Since μ is a (finite) Borel measure and $\mu \neq 0$ there is a set E δ $[-\pi, \pi]$ which is finite or countably infinite such that $\mu(\{e^{i\theta}\}) = 0$ if θ ⌠ E. This proves the first part of the theorem.

Conversely, let E be a finite or countably infinite set and let E = $\{\theta_n$: n 0 N$\}$ where N is a set of positive integers and $\theta_n \neq \theta_m$ for $n \neq m$. Define a sequence $\{\lambda_n\}$ (n 0 N) such that $\lambda_n \neq 0$ and $\sum_{n\in N} |\lambda_n| < \infty$. Let μ be the measure on T that is supported on $\{e^{i\theta}$: θ 0 E$\}$ and has mass λ_n at $e^{i\theta_n}$ for each n 0 N. Let $f(z) = \int_T \frac{1}{(1-\bar{\zeta}z)^{\alpha}} d\mu(\zeta)$ for $|z| < 1$. The argument given above implies that $\lim_{\substack{z\to e^{i\theta_n}\\ z\in S}} (1-e^{-i\theta_n}z)^{\alpha} f(z) = \mu\Big(\{e^{i\theta_n}\}\Big) = \lambda_n \neq 0$ for every n 0 N and for every Stolz angle S with vertex $e^{i\theta_n}$. Also, if θ ⌠ E and S is a Stolz angle with vertex $e^{i\theta}$ then

$$\lim_{\substack{z\to e^{i\theta}\\ z\in S}} (1-e^{-i\theta}z)^{\alpha} f(z) = \mu\Big(\{e^{i\theta}\}\Big) = 0.$$

Next we show that the maximal growth for a function in $\mathcal{F}_\alpha$ is reduced by 1 when $\alpha > 1$ and when the exceptional sets have Lebesgue measure zero.

Theorem 4.11 *If* $\alpha > 1$ *and f* 0 $\mathcal{F}_\alpha$ *then the nontangential limit of* $(1 - e^{-i\theta}z)^{\alpha-1}f(z)$ *at* $e^{i\theta}$ *exists and equals zero for almost all* θ *in* $[-\pi, \pi]$.

Theorem 4.11 is a consequence of Theorem 4.12 stated below and the fact that each function of bounded variation on $[-\pi, \pi]$ is differentiable almost everywhere on $[-\pi, \pi]$.

Theorem 4.12 *Suppose that g is a complex-valued function of bounded variation on* $[-\pi, \pi]$ *and g is differentiable at some* θ *where* $-\pi \leq \theta \leq \pi$. *Let the function f be defined by*

$$f(z) = \int_{-\pi}^{\pi} \frac{1}{(1-e^{-it}z)^{\alpha}} dg(t) \qquad (|z| < 1). \tag{4.31}$$

If $\alpha > 1$ *then* $(1 - e^{-i\theta}z)^{\alpha-1} f(z)$ *has the nontangential limit* 0 *at* $e^{i\theta}$.

Proof: Since $\displaystyle\int_{\theta-\pi}^{\theta+\pi} \frac{1}{(1-e^{-it}z)^{\alpha}}\,dt = 2\pi$, (4.31) and periodicity give

$$f(z) = \int_{\theta-\pi}^{\theta+\pi} \frac{1}{(1-e^{-it}z)^{\alpha}}\, d\big[g(t) - g(\theta) - (t-\theta)g'(\theta)\big] + 2\pi g'(\theta)$$

$$= \int_{-\pi}^{\pi} \frac{1}{(1-e^{-i(\theta+t)}z)^{\alpha}}\, d\big[g(\theta+t) - g(\theta) - t g'(\theta)\big] + 2\pi g'(\theta).$$

An integration by parts yields

$$\begin{aligned} f(z) &= 2\pi g'(\theta) + \frac{g(\theta+\pi) - 2\pi g'(\theta) - g(\theta-\pi)}{(1-e^{-i\theta}z)^{\alpha}} \\ &\quad + i\alpha \int_{-\pi}^{\pi} K\left(ze^{-i(\theta+t)}\right)\big[g(\theta+t) - g(\theta) - t g'(\theta)\big]\,dt \end{aligned} \tag{4.32}$$

where $K(z) = \dfrac{z}{(1-z)^{\alpha+1}}$.

Suppose that $0 < \gamma < \pi$ and let $S = S(\theta, \gamma)$. Then (4.32) implies that

$$|f(z)| \le A + \alpha \int_{-\pi}^{\pi} \frac{|g(\theta+t) - g(\theta) - t g'(\theta)|}{\left|1 - e^{-i(\theta+t)}z\right|^{\alpha+1}}\,dt \tag{4.33}$$

for z 0 S where A is a positive constant. Since g is differentiable at θ,

$$g(\theta+t) = g(\theta) + t g'(\theta) + t\,h(t)$$

where $\lim_{t\to 0} h(t) = 0$. Suppose that $\varepsilon > 0$. There exists a real number δ such that $0 < \delta < 1$ and $|h(t)| < \varepsilon$ for $|t| < \delta$. There are positive constants B and C such that if $z = re^{i\varphi}$ 0 S for suitable φ then $|z - e^{i\theta}| \le B(1 - |z|)$ and $|\varphi - \theta| \le C(1 - |z|)$. Hence there exists a real number η with $0 < \eta \le \frac{1}{2}$ and such that if z 0 S and $|z - e^{i\theta}| < \eta$ then $2C(1 - |z|) < \delta$.

For $-\pi \le t \le \pi$ and $|z| < 1$ let

$$G(t,z) = \frac{|g(\theta+t) - g(\theta) - tg'(\theta)|}{\left|1-e^{i(\theta+t)}z\right|^{\alpha+1}}.$$

Then (4.33) yields

$$|f(z)| \le A + \alpha \sum_{n=1}^{5} J_n \tag{4.34}$$

where

$$J_n = \int_{I_n} G(t,z)\,dt$$

and $I_1 = [-\pi, -\delta]$, $I_2 = [-\delta, -2C(1-|z|)]$, $I_3 = [-2C(1-|z|), 2C(1-|z|)]$,

$I_4 = [2C(1-|z|), \delta]$ and $I_5 = [\delta, \pi]$.

There are positive constants D and E such that $J_1 \le D$ and $J_5 \le E$ for z 0 S.

Suppose that z 0 S and $z = re^{i\varphi}$ for suitable φ where $0 \le r < 1$. Then $\eta \le ½$ implies that $r \ge ½$ and hence $\left|1 - re^{it}\right|^2 \ge (1-r)^2 + \frac{2}{\pi^2} t^2$. Hence

$$J_2 \le \left(\frac{\pi}{\sqrt{2}}\right)^{\alpha+1} \varepsilon \int_{-\delta}^{-2C(1-r)} \frac{|t|}{|t-(\varphi-\theta)|^{\alpha+1}}\,dt.$$

If we let $Q(t) = \frac{|t|}{|t-(\varphi-\theta)|}$ where $-\delta \le t \le -2C(1-r)$ and $|\varphi-\theta| \le C(1-r)$, it follows that $Q(t) \le 2$. Hence

$$J_2 \le \frac{\pi^{\alpha+1}}{2^{(\alpha-1)/2}} \varepsilon \int_{-\delta}^{-2C(1-r)} \frac{1}{|t-(\varphi-\theta)|^{\alpha}}\,dt.$$

Since $|\varphi-\theta| \le C(1-r)$ this yields

$$J_2 \le \frac{\pi^{\alpha+1}}{2^{(\alpha-1)/2}}\ \varepsilon \int_{2C(1-r)}^{\delta} \frac{1}{[s+(\varphi-\theta)]^{\alpha}}\, ds$$

$$\le \frac{\pi^{\alpha+1}}{2^{(\alpha-1)/2}}\ \frac{\varepsilon}{\alpha-1}\ \{2C(1-r)+(\varphi-\theta)\}^{1-\alpha}$$

$$\le \frac{\pi^{\alpha+1}}{2^{(\alpha-1)/2}}\ \frac{\varepsilon}{\alpha-1}\ \frac{1}{\{C(1-r)\}^{\alpha-1}}.$$

A similar argument yields the same inequality for J_4. Also

$$J_3 \le \int_{-2C(1-r)}^{2C(1-r)} \frac{\varepsilon\,|t|}{(1-r)^{\alpha+1}}\, dt = \frac{4C^2\varepsilon}{(1-r)^{\alpha-1}}$$

Since $|z-e^{i\theta}| \le B(1-r)$, the estimates on J_n and (4.34) imply that

$$|(1-e^{-i\theta}z)^{\alpha-1}\ f(z)| \le F(1-r)^{\alpha-1} + G\varepsilon$$

for z 0 S and $|z-e^{i\theta}| < \eta$ where F and G are positive constants. Therefore

$$\lim_{\substack{z\to e^{i\theta}\\ z\in S}} \left((1-e^{-i\theta}z)^{\alpha-1}\ f(z)\right) = 0.$$

Theorem 4.12 is sharp, at least when $1 < \alpha \le 2$, in the following sense. Suppose that $1 < \alpha \le 2$ and ε is a positive nonincreasing function on (0, 1) such that $\lim_{r\to 1-} \varepsilon(r) = 0$. Then there is a function g of bounded variation on $[-\pi, \pi]$ which is differentiable at 0 such that if f is the function defined by (4.31) then $\overline{\lim_{r\to 1-}}\ \frac{|f(r)|\,(1-r)^{\alpha-1}}{\varepsilon(r)} = \infty$. A reference for this result is in Hallenbeck and MacGregor [1993c], which also includes the development of Theorem 4.13 and Lemmas 4.14 and 4.15.

The next theorem concerns functions in $\mathcal{F}_\alpha$ with radial growth between the two growths discussed in Theorems 4.10 and 4.11. Now the exceptional sets are described in terms of capacity.

Theorem 4.13 *Suppose that* $\alpha > 0$ *and* f 0 $\mathcal{F}_\alpha$. *If* $0 < \beta < 1$ *and* $\beta < \alpha$ *then* $(1-e^{-i\theta}z)^{\alpha-\beta} f(z)$ *has the nontangential limit zero at* $e^{i\theta}$ β *quasi-everywhere.*

Theorem 4.13 is a consequence of the following lemmas.

Lemma 4.14 L*et* $\alpha > 0$ *and for* $|z| < 1$ *let* $f(z) = \int_{-\pi}^{\pi} \frac{1}{(1-e^{-it}z)^\alpha}\, dg(t)$ *where the function* g *is of bounded variation on* $[-\pi, \pi]$. *Suppose that* $\beta > 0$ *and suppose that for some* θ *in* $[-\pi, \pi]$

$$|g(t) - g(\theta)| = o(|t-\theta|^\beta)$$

as $t \to \theta$. *If* $\beta < \alpha$, *then* $(1 - e^{-i\theta}z)^{\alpha-\beta} f(z)$ *has the nontangential limit zero at* $e^{i\theta}$.

Lemma 4.15 S*uppose that* g *is a real-valued nondecreasing function on* $[-\pi, \pi]$. *If* $0 < \beta < 1$, *then the relation*

$$|g(t) - g(\theta)| = o(|t-\theta|^\beta) \text{ as } t \to \theta$$

holds β *quasi-everywhere in* θ.

NOTES

In Koosis [1980; see p. 129] Lemma 1 is given as part of an argument to prove a theorem about harmonic conjugates due to Kolmogorov. Theorems 2 and 3 are in MacGregor [2004]. Lemma 4 is a classical result of Fatou [1906]. Lemma 5 and Theorems 6 and 9 were proved by Hallenbeck and MacGregor [1993b]. References about α-capacity are Hayman and Kennedy [1976], Landkof [1972] and Tsuji [1959]. Lemma 8 was proved by Twomey [1988]. When $\alpha = 0$, a result stronger than Theorem 9 was proved by Hallenbeck and Samotij [1993]; namely, if f 0 $\mathcal{F}_0$ then there is a set $\Phi \subset [-\pi, \pi]$ such that the radial variation of f in the direction $e^{i\theta}$ is finite for every θ 0 $[-\pi, \pi] \setminus \Phi$ and the logarithmic capacity of Φ is zero. Theorem 10 is in Hallenbeck and MacGregor [1993a]. Theorems 11, 12 and 13 and Lemmas 14 and 15 were proved by Hallenbeck and MacGregor [1993c]. MacGregor [1995] gives a survey of results about radial limits of fractional Cauchy transforms.

We have focused our discussion on nontangential limits of functions in $\mathcal{F}_\alpha$. Hallenbeck [1997] studies tangential limits and includes a proof that if f 0 $\mathcal{F}_\alpha$ and $0 < \alpha < 1$ then f has a limit under a certain tangential approach to $e^{i\theta}$, depending on α, for almost all θ.

CHAPTER 5

Zeros

Preamble. We study the problem of describing the zeros of functions in $\mathcal{F}_\alpha$.

If $f \in \mathcal{F}_\alpha$ for some α with $0 \le \alpha \le 1$, then f belongs to certain Hardy spaces. Hence, if $f \ne 0$ then the zeros $\{z_n\}$ of f satisfy the Blaschke condition $\sum_n (1-|z_n|) < \infty$. Conversely, if a sequence $\{z_n\}$ in $\mathcal{D}$ satisfies the Blaschke condition, then the Blaschke product with those zeros is well defined and belongs to H^∞, and hence to $\mathcal{F}_1$. Therefore the Blaschke condition characterizes the zeros of functions $f \in \mathcal{F}_1$ with $f \ne 0$. This condition also characterizes the zeros of $f \in \mathcal{F}_\alpha$ for each $\alpha \ge 0$, if the zeros lie in some Stolz angle or in a finite union of Stolz angles. This is the content of Corollary 5.3 and Theorem 5.4.

In general, the Blaschke condition does not characterize the zeros of functions $f \ne 0$ in $\mathcal{F}_\alpha$ when $\alpha \ne 1$. Little is known when $0 \le \alpha < 1$. The case $\alpha > 1$ is well understood. Theorem 5.1 shows that if $\alpha > 1$, $f \in \mathcal{F}_\alpha$ and $f \ne 0$, then the moduli of the zeros of f satisfy a certain concrete condition. This condition is less restrictive than the Blaschke condition. A later result shows that this condition is sharp in a strong way. The argument relies on the construction of a suitable lacunary series. As a consequence, the result in Theorem 3.8 about the growth of the integral means of a function $f \in \mathcal{F}_\alpha$ ($\alpha > 1$) is shown to be sharp. This yields the further result that if $\alpha > 1$ then there is a function $f \in \mathcal{F}_\alpha$ such that $\overline{\lim_{r \to 1-}} |f(re^{i\theta})| = \infty$ for every θ in $[-\pi, \pi]$.

Chapter 8 includes results about the factorization of functions in $\mathcal{F}_\alpha$ in terms of their zeros.

We consider the problem of describing the zeros of functions in $\mathcal{F}_\alpha$. If N is a positive integer and $z_n \in \mathcal{D}$ for $n = 1, 2, \ldots, N$ then the polynomial $\prod_{n=1}^{N} (z - z_n)$

is analytic in $\overline{D}$ and hence belongs to $\mathcal{F}_\alpha$ for all $\alpha \geq 0$. Thus any finite sequence is a permissible zero sequence for each family $\mathcal{F}_\alpha$.

Let $\{z_n\}$ (n = 1, 2, …) be an infinite sequence in D with $z_n \neq 0$ for all n. We seek a nonconstant analytic function vanishing at each number in the sequence $\{z_n\}$, with multiplicity given by how often that number occurs in sequence. Thus we assume that $\{z_n\}$ has no point of accumulation in D. Also we assume that $\{|z_n|\}$ is nondecreasing. Finally we assume that the power series expansion for f at z = 0 is

$$f(z) = bz^m + \cdots \qquad (|z| < 1) \tag{5.1}$$

where $b \neq 0$ and m is a nonnegative integer.

We describe one general approach for obtaining information about the zeros of a function. Suppose that the function f is analytic in D and $f \neq 0$. Then $\log|f|$ is subharmonic in D and the function L defined by

$$L(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\, d\theta \qquad (0 < r < 1) \tag{5.2}$$

is nondecreasing on (0, 1). If f has the expansion (5.1), $0 < r < 1$ and the zeros of f in $\{z: 0 < |z| < r\}$, with due count of multiplicities, are given by $\{z_n: n = 1, 2, \ldots, p\}$, then Jensen's formula gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\, d\theta = \sum_{n=1}^{p} \log\frac{r}{|z_n|} + \log|b| + m\log r. \tag{5.3}$$

It is a consequence of (5.3) that L is bounded above if and only if

$$\sum_{n=1}^{\infty}\left(1 - |z_n|\right) < \infty. \tag{5.4}$$

Since $\log x \leq \frac{1}{p}x^p$ for $p > 0$ and $x \geq 0$, the previous remarks imply that if f 0 H^p for some $p > 0$ then (5.4) follows. More generally, (5.4) holds for the zeros of a function in the Nevanlinna class *N*. A function f belongs to *N* provided that f is analytic in D and

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \log^{+} |f(re^{i\theta})|\, d\theta < \infty \tag{5.5}$$

where

$$\log^{+} x = \begin{cases} \log x & \text{if } x \geq 1 \\ 0 & \text{if } 0 < x < 1. \end{cases}$$

Since $\log x \leq \log^{+} x$ for $x \geq 0$ it follows that if f 0 N then L is bounded above and this yields (5.4).

Suppose that $f \neq 0$ and the zeros of f in $\{z: 0 < |z| < 1\}$ are given by $\{z_n\}$ (n = 1, 2, …). Further assume that f 0 $\mathcal{F}_\alpha$ for some α, $0 \leq \alpha \leq 1$. Then f 0 H^p for suitable p and therefore (5.4) follows. The condition (5.4) characterizes the zero sequences of nonconstant functions in $\mathcal{F}_1$. This follows from the fact that (5.4) implies that the Blaschke product B with zeros $\{z_n\}$ is well-defined and bounded in $\mathcal{D}$. Hence B 0 $\mathcal{F}_1$.

The following theorem gives information about the zeros of functions in $\mathcal{F}_\alpha$ when $\alpha > 1$. Note that if $0 < |z_n| < 1$ and $|z_n| \to 1$, then (5.4) is equivalent to the convergence of $\prod_{k=1}^{\infty} \frac{1}{|z_k|}$. Hence (5.4) implies that the condition (5.6) in the next theorem holds for every $\alpha > 1$. In general (5.6) is less restrictive than (5.4).

Theorem 5.1 *Suppose that* $\alpha > 1$, f 0 $\mathcal{F}_\alpha$ and $f \neq 0$. *Let the zeros of* f *in* $\{z: 0 < |z| < 1\}$ *be given by* $\{z_n\}$ (n = 1, 2, …), *with due count of multiplicities, and assume that* $\{|z_n|\}$ *is nondecreasing. Then*

$$\lim_{n\to\infty} \frac{1}{n^{\alpha-1}} \prod_{k=1}^{n} \frac{1}{|z_k|} = 0. \tag{5.6}$$

Proof: For $0 < r < 1$ let

$$\varepsilon(r) = (1-r)^{\alpha-1} M_0(r, f) = (1-r)^{\alpha-1} \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})|\, d\theta \right\}.$$

Then the function L defined by (5.2) is given by

$$L(r) = \log \left\{ \frac{\varepsilon(r)}{(1-r)^{\alpha-1}} \right\}. \tag{5.7}$$

For $0 < r < 1$ let $n(r)$ denote the number of terms from the sequence $\{z_k\}$ which belong to $\{z: |z| < r\}$. Suppose that n is a positive integer and $n \le n(r)$. Then $|z_k| \le r$ for $k = n + 1, n + 2, \ldots, n(r)$ and hence

$$\prod_{k=1}^{n(r)} \frac{r}{|z_k|} = \prod_{k=1}^{n} \frac{r}{|z_k|} \prod_{k=n+1}^{n(r)} \frac{r}{|z_k|} \ge \prod_{k=1}^{n} \frac{r}{|z_k|}.$$

Next suppose that n is a positive integer and $n > n(r)$. Then $|z_k| > r$ for $k = n(r) + 1, n(r) + 2, \ldots, n$ and hence

$$\prod_{k=1}^{n} \frac{r}{|z_k|} = \prod_{k=1}^{n(r)} \frac{r}{|z_k|} \prod_{k=n(r)+1}^{n} \frac{r}{|z_k|} \le \prod_{k=1}^{n(r)} \frac{r}{|z_k|}.$$

Therefore

$$\prod_{k=1}^{n} \frac{r}{|z_k|} \le \prod_{k=1}^{n(r)} \frac{r}{|z_k|} \tag{5.8}$$

for $n = 1, 2, \ldots$.

Suppose that f has the form (5.1). If we use (5.3) with $p = n(r)$ and (5.7), we obtain

$$\prod_{k=1}^{n(r)} \frac{r}{|z_k|} = \frac{\varepsilon(r)}{|b| r^m (1-r)^{\alpha-1}}. \tag{5.9}$$

Thus (5.8) yields

$$\prod_{k=1}^{n} \frac{r}{|z_k|} \le \frac{\varepsilon(r)}{|b| r^m (1-r)^{\alpha-1}} \tag{5.10}$$

for $0 < r < 1$ and $n = 1, 2, \ldots$. For each n set $r = \dfrac{n}{n+1}$ in (5.10). This shows that

$$\frac{1}{(n+1)^{\alpha-1}} \prod_{k=1}^{n} \frac{1}{|z_k|} \le \delta_n \tag{5.11}$$

where

$$\delta_n = \varepsilon\left(\frac{n}{n+1}\right) \bigg/ \left(\frac{n}{n+1}\right)^{m+n} |b|$$

for n = 1, 2, … . Corollary 3.9 yields $\lim_{r\to 1-} \varepsilon(r) = 0$ and hence $\lim_{n\to\infty} \delta_n = 0$. Therefore (5.11) implies (5.6). □

Later we show that when $\alpha > 1$ the condition (5.6) gives precise information about the sequence of zeros of nonconstant functions in $\mathcal{F}_\alpha$. The problem of characterizing the zeros of functions in $\mathcal{F}_\alpha$ when $0 \le \alpha < 1$ is unresolved. In Theorem 2.16 we showed that if

$$\sum_{n=1}^{\infty} (1 - |z_n|)^\alpha < \infty \tag{5.12}$$

and $0 < \alpha < 1$, then the Blaschke product with zeros $\{z_n\}$ belongs to $\mathcal{F}_\alpha$. Also it is known that if $0 < \alpha < 1$, $0 < r_n < 1$ for n = 1, 2, … and $\sum_{n=1}^{\infty} (1 - r_n)^\alpha = \infty$, then there are numbers θ_n in $[-\pi, \pi]$ such that if $z_n = r_n e^{i\theta_n}$, $f \in \mathcal{F}_\alpha$ and $f(z_n) = 0$ for n = 1, 2, … , then $f = 0$ (see Nagel, Rudin and J.H. Shapiro [1982]; p. 359). In particular, the condition (5.4) does not characterize zero sequences of nonconstant functions in $\mathcal{F}_\alpha$ when $0 < \alpha < 1$.

We next consider the situation where the zeros of a function in $\mathcal{F}_\alpha$ belong to some Stolz angle in $\mathbb{D}$. We find that in this case, (5.4) gives a complete description of the zeros of a nonconstant function in $\mathcal{F}_\alpha$ for all $\alpha \ge 0$. This is a consequence of more general results given below.

Theorem 5.2 *Suppose that the function f is analytic in* $\mathbb{D}$, $f \neq 0$ *and there are real constants* A and β *such that* $A > 0$, $0 < \beta < ½$ *and*

$$|f(z)| \le A \exp\left[\frac{1}{(1-|z|)^\beta}\right] \tag{5.13}$$

for $|z| < 1$. *Let* $\{z_n\}$ (n = 1, 2, …) *denote the zeros of f listed according to multiplicities. If there is a Stolz angle* S *with* $z_n \in S$ *for* n = 1, 2, … , *then*

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty. \tag{5.14}$$

Proof: We may assume that the Stolz angle S has vertex 1. There is a positive constant B such that

$$|1 - z| \leq B\,(1 - |z|) \tag{5.15}$$

for z 0 S. Inequality (5.13) implies that there is a positive constant C such that

$$\log^+ |f(z)| \leq \frac{C}{(1 - |z|)^\beta} \tag{5.16}$$

for $|z| < 1$. Let $\Omega = \{z: |z - ½\,| < ½\,\}$. Let z 0 Ω and set $z = ½ + re^{i\theta}$ where $-\pi \leq \theta \leq \pi$ and $0 \leq r < ½$. Then $1 - |z|^2 \geq ½\,(1 - \cos\theta)$ for $|\theta| \leq 2\pi/3$. Hence, if $0 < |\theta| \leq 2\pi/3$ then $\dfrac{\theta^2}{1-|z|} \leq \dfrac{4\theta^2}{1-\cos\theta}$. Since $\lim\limits_{\theta\to 0} \dfrac{\theta^2}{1-\cos\theta}$ exists, this shows that there is a positive constant D such that

$$\frac{\theta^2}{1-|z|} \leq D \tag{5.17}$$

for $z = ½ + re^{i\theta}$ where $0 < r < ½$ and $-\pi \leq \theta \leq \pi$.

Let the function g be defined by $g(w) = f(z)$ where $z = \dfrac{1+w}{2}$ and $|w| < 1$. Set $w = \rho e^{i\theta}$ where $0 < \rho < 1$ and $-\pi \leq \theta \leq \pi$. Then (5.16) and (5.17) yield

$$\log^+ \left|g(\rho e^{i\theta})\right| = \log^+\left|f(z)\right| \leq \frac{E}{\theta^{2\beta}}$$

where $E = CD^\beta$. Since $0 < \beta < ½$ the integral $\displaystyle\int_{-\pi}^{\pi} \frac{1}{\theta^{2\beta}}\, d\theta$ is finite, and hence there is a positive constant F such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ \left|g(\rho e^{i\theta})\right| d\theta \leq F$$

for $0 < \rho < 1$. Therefore g 0 N.

Since z_n 0 S and $z_n \to 1$ there is an integer J such that z_n 0 Ω for $n \geq J$. Let $w_n = 2z_n - 1$ for n = 1, 2, … . Then $|w_n| < 1$ for $n \geq J$. Because g 0 N and

$g(w_n) = 0$ we conclude that

$$\sum_{n=J}^{\infty}\left(1 - |w_n|\right) < \infty. \qquad (5.18)$$

Suppose that z 0 S 1 Ω and let w = 2z – 1. If z is sufficiently close to 1, then w 0 S and (5.15) yields

$$2(1 - |z|) \le 2\,|1 - z| = |1 - w| \le B(1 - |w|).$$

Hence there is an integer K such that K ≥ J and a positive constant G such that

$$1 - |z_n| \le G\,(1 - |w_n|) \qquad (5.19)$$

for n ≥ K. Inequalities (5.18) and (5.19) imply (5.14). □

The argument given for Theorem 5.2 can be used to show that (5.14) holds more generally under the assumption (5.13) where 0 < β < 1. This depends on replacing Ω by a domain Φ ⊂ D which has a lower order of contact with ∂D at 1, and replacing w = 2z – 1 by a conformal mapping of Φ onto D. In the definitive result of this type, (5.13) is replaced by $|f(z)| \le \exp[M(r)]$ for $|z| \le r$, where $M(r) \ge 0$, $\lim_{r \to 1-} M(r) = \infty$ and $\int_0^1 \left[\frac{M(r)}{1-r}\right]^{1/2} dr < \infty$ (see Hayman and Korenblum [1980]).

Corollary 5.3 *Suppose that f* 0 F$_\alpha$ *for some* α ≥ 0, *f* ≠ 0 and *f* (z_n) = 0 *for* n = 1, 2, … ($|z_n| < 1$). *If there is a Stolz angle* S *such that* z_n 0 S *for all* n, *then* $\sum_{n=1}^{\infty}\left(1 - |z_n|\right) < \infty$.

Proof: As noted earlier, the condition $\sum_{n=1}^{\infty}\left(1 - |z_n|\right) < \infty$ holds when 0 ≤ α ≤ 1 whether or not the zeros lie in a Stolz angle. For the general case, recall that if *f* 0 F$_\alpha$ where α > 0 then $\left|f(z)\right| \le \frac{B}{(1-|z|)^{\alpha}}$ for $|z| < 1$, where B is a positive constant. This inequality implies that (5.13) holds for every β > 0. An application of Theorem 5.2 for some β with 0 < β < ½ yields the corollary. □

Theorem 5.4 *Suppose that* m *is a nonnegative integer and* $\{z_n\}$ (n = 1, 2, …) *is a sequence of complex numbers with* $0 < |z_n| < 1$. *Further assume that there is a Stolz angle* S *with* z_n 0 S (n = 1, 2, …) *and* $\sum_{n=1}^{\infty} (1 - |z_n|) < \infty$. *Then there exists a function f belonging to* $\mathcal{F}_\alpha$ *for all* $\alpha \geq 0$ *such that f has a zero of order* m *at zero and the remaining zeros of f are given by* $\{z_n\}$.

Proof: We may assume that the Stolz angle S has vertex 1. There is a positive constant A such that $|1 - z| \leq A\,(1 - |z|)$ for z 0 S. Hence

$$|1 - z_n| \leq A\,(1 - |z_n|). \tag{5.20}$$

for n = 1, 2, … . Let

$$g(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z} \qquad (|z| < 1).$$

Then

$$g'(z) = \sum_{n=1}^{\infty} g_n(z) \frac{|z_n|}{z_n} \frac{|z_n|^2 - 1}{(1 - \bar{z}_n z)^2} \tag{5.21}$$

where

$$g_n(z) = \prod_{k \neq n} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \bar{z}_k z}.$$

Since $|g_n(z)| \leq 1$ this yields

$$|g'(z)| \leq \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|1 - \bar{z}_n z|^2}.$$

Inequality (5.20) implies that $\left| \frac{1 - z}{1 - \bar{z}_n z} \right| \leq A + 1$ for $|z| < 1$. Therefore

$$|g'(z)| \leq \frac{2(A + 1)^2}{|1 - z|^2} \sum_{n=1}^{\infty} (1 - |z_n|) = \frac{B}{|1 - z|^2}$$

where B is a positive constant.

Let $h(z) = (1 - z)^2\, g(z)$ for $|z| < 1$. Then $h'(z) = (1 - z)^2\, g'(z) - 2\,(1 - z)\, g(z)$ and hence $|h'(z)| \le B + 4$ for $|z| < 1$. Let $f(z) = z^m\, h(z)$ for $|z| < 1$. Then f is analytic in $\mathcal{D}$ and f has the required zeros. Also $|f'(z)| \le |h'(z)| + m\,|h(z)|$ for $|z| < 1$. Since h' is bounded, it follows that h is bounded and therefore f' is bounded. Hence f' 0 $\mathcal{F}_1$ and then f 0 $\mathcal{F}_0$ by Theorem 2.8. Theorem 2.10 yields f 0 $\mathcal{F}_\alpha$ for all $\alpha \ge 0$. □

The next theorem provides a construction of a suitable function in $\mathcal{F}_\alpha$ for $\alpha > 1$.

Theorem 5.5 *Suppose that* $\alpha > 1$ *and* ε *is a positive function defined on* $(0,1)$ *such that* $\lim_{r\to 1-} \varepsilon(r) = 0$. *Then there is a function* f *in* $\mathcal{F}_\alpha$ *such that*

$$\overline{\lim_{r\to 1-}} \left\{ \frac{(1-r)^{\alpha-1}}{\varepsilon(r)} \min_{|z|=r} |f(z)| \right\} = \infty \;. \tag{5.22}$$

Furthermore,

$$\overline{\lim_{r\to 1-}} \left\{ \frac{(1-r)^{\alpha-1}}{\varepsilon(r)} M_p(r, f) \right\} = \infty \tag{5.23}$$

for every $p > 0$, *and*

$$\overline{\lim_{r\to 1-}} \left\{ \frac{(1-r)^{\alpha-1}}{\varepsilon(r)} M_0(r, f) \right\} = \infty. \tag{5.24}$$

Proof: Since $M_p(r, f) \ge \min_{|z|=r} |f(z)|$ for $0 < r < 1$ and $p \ge 0$, it follows that (5.22) implies (5.23) and (5.24). Hence it suffices to prove (5.22).

Suppose that $\alpha > 1$ and the function ε obeys the hypotheses of the theorem. To prove (5.22) it suffices to show that there is a function f in $\mathcal{F}_\alpha$ with the following property: there is a positive constant A and a sequence $\{r_k\}$ $(k = 1, 2, \ldots)$ such that $0 < r_k < 1$ for $k = 1, 2, \ldots$, $r_k \to 1$ as $k \to \infty$ and

$$(1 - r_k)^{\alpha-1} \left| f(r_k e^{i\theta}) \right| \ge A\, \varepsilon(r_k) \tag{5.25}$$

for all θ in $[-\pi, \pi]$ and for all large k. This follows by first obtaining f such that (5.25) holds with ε replaced by $\sqrt{\varepsilon}$, which then yields (5.22).

Let $\{\lambda_k\}$ $(k = 1, 2, \ldots)$ be an increasing sequence of integers with $\lambda_1 \geq 2$ and

$$\lambda_{k+1} \geq (k+1)^{3/(\alpha-1)} \lambda_k \tag{5.26}$$

and

$$\varepsilon\left(1 - \frac{1}{\lambda_k}\right) \leq \frac{1}{k^2} \tag{5.27}$$

for $k = 1, 2, \ldots$. Such a sequence can be defined inductively using $\alpha > 1$ and the assumption $\lim_{r \to 1-} \varepsilon(r) = 0$. Since $\alpha > 1$, (5.26) gives

$$\frac{\lambda_{k+1}{}^{\alpha-1}}{(k+1)^3} \geq \lambda_k{}^{\alpha-1} .$$

For $n = 1, 2, \ldots, k$ we have $\lambda_k{}^{\alpha-1} \geq \lambda_n{}^{\alpha-1} \geq \lambda_n{}^{\alpha-1}/n^2$, and therefore

$$\frac{\lambda_{k+1}{}^{\alpha-1}}{(k+1)^3} \geq \frac{\lambda_n{}^{\alpha-1}}{n^2} \tag{5.28}$$

for $n = 1, 2, \ldots, k$ and for every $k = 1, 2, \ldots$.

Let the functions g and f be defined by

$$g(z) = \sum_{n=1}^{\infty} \frac{1}{n^2} z^{\lambda_n} \tag{5.29}$$

and

$$f(z) = \sum_{n=1}^{\infty} \frac{\lambda_n{}^{\alpha-1}}{n^2} z^{\lambda_n} \tag{5.30}$$

for $|z| < 1$. Clearly g and f are analytic in $\mathcal{D}$ and g is bounded. Hence $g \in \mathcal{F}_1$. Previous arguments about the asymptotic expansion for $A_n(\alpha)$ and the fact that $g \in \mathcal{F}_1$ yield $f \in \mathcal{F}_\alpha$. For $k = 1, 2, \ldots$ let

$$r_k = 1 - \frac{1}{\lambda_k}. \tag{5.31}$$

Let the functions P_k, Q_k and R_k be defined by

$$P_k(z) = \sum_{n=1}^{k-1} \frac{\lambda_n{}^{\alpha-1}}{n^2} z^{\lambda_n}, \tag{5.32}$$

$$Q_k(z) = \frac{\lambda_k^{\alpha-1}}{k^2} z^{\lambda_k}, \tag{5.33}$$

and

$$R_k(z) = \sum_{n=k+1}^{\infty} \frac{\lambda_n{}^{\alpha-1}}{n^2} z^{\lambda_n} \tag{5.34}$$

for $|z| < 1$. Then

$$f = P_k + Q_k + R_k. \tag{5.35}$$

Suppose that $k \geq 3$. Then (5.32) and (5.28) imply that for $-\pi \leq \theta \leq \pi$ we have

$$\begin{aligned} \left|P_k(r_k e^{i\theta})\right| &\leq \sum_{n=1}^{k-2} \frac{\lambda_n{}^{\alpha-1}}{n^2} + \frac{\lambda_{k-1}{}^{\alpha-1}}{(k-1)^2} \\ &\leq \sum_{n=1}^{k-2} \frac{\lambda_{k-1}{}^{\alpha-1}}{(k-1)^3} + \frac{\lambda_{k-1}{}^{\alpha-1}}{(k-1)^2}. \end{aligned}$$

Thus

$$\left|P_k(r_k e^{i\theta})\right| \leq \frac{2k-3}{(k-1)^3} \lambda_{k-1}{}^{\alpha-1} \tag{5.36}$$

for $k \geq 3$ and $-\pi \leq \theta \leq \pi$. Also

$$\left|Q_k(r_k e^{i\theta})\right| = \frac{\lambda_k{}^{\alpha-1}}{k^2} r_k{}^{\lambda_k} \tag{5.37}$$

for $k \geq 1$ and $-\pi \leq \theta \leq \pi$.

For each $k \geq 3$ define the function U_k by $U_k(\theta) = \dfrac{P_k(r_k e^{i\theta})}{Q_k(r_k e^{i\theta})}$ where $-\pi \leq \theta \leq \pi$. Then (5.36) and (5.37) yield

$$\left| U_k(\theta) \right| \leq \frac{2k^3 - 3k^2}{(k-1)^3} \left(\frac{\lambda_{k-1}}{\lambda_k} \right)^{\alpha-1} \frac{1}{r_k^{\lambda_k}}$$

for $-\pi \leq \theta \leq \pi$ and $k \geq 3$. The relation (5.31) yields $r_k^{\lambda_k} \to 1/e$. Also (5.26) implies that $\dfrac{\lambda_{k-1}}{\lambda_k} \to 0$. Therefore the previous inequality shows that $U_k \to 0$ uniformly on $[-\pi, \pi]$. Hence

$$\left| P_k(r_k e^{i\theta}) \right| \leq \frac{1}{4} \left| Q_k(r_k e^{i\theta}) \right| \qquad (5.38)$$

for $-\pi \leq \theta \leq \pi$ and for all large k.

Let $\beta = \alpha - 1$. Then (5.26) implies that $\lambda_{k+1} \geq 2\beta\, \lambda_k$ for all large k, and thus $\beta/(1-r_k) = \beta\, \lambda_k < \lambda_{k+1}$. Hence $\beta/(1-r_k) < \lambda_k$ for $n \geq k+1$ and for all large k. For a fixed r $(0 < r < 1)$, the function $x \mapsto x^{\beta} r^{x}$ defined for $x > 0$ has its maximum at $\dfrac{-\beta}{\log r} = \dfrac{\beta}{1-r} + O(1)$ as $r \to 1-$. Therefore

$$\lambda_n^{\alpha-1} r_k^{\lambda_n} \leq \lambda_{k+1}^{\alpha-1} r_k^{\lambda_{k+1}}$$

for $n \geq k+1$ when k is sufficiently large. This inequality and (5.34) yield

$$\begin{aligned}
\left| R_k(r_k e^{i\theta}) \right| &\leq \sum_{n=k+1}^{\infty} \frac{\lambda_n^{\alpha-1}}{n^2} r_k^{\lambda_n} \\
&\leq \lambda_{k+1}^{\alpha-1} r_k^{\lambda_{k+1}} \sum_{n=k+1}^{\infty} \frac{1}{n^2} \\
&\leq \lambda_{k+1}^{\alpha-1} r_k^{\lambda_{k+1}} \int_k^{\infty} \frac{1}{x^2}\, dx \\
&= \lambda_{k+1}^{\alpha-1} r_k^{\lambda_{k+1}} \frac{1}{k}.
\end{aligned}$$

Hence (5.26) and (5.37) give

$$\left|\frac{R_k(r_k e^{i\theta})}{Q_k(r_k e^{i\theta})}\right| \le \frac{1}{r_k^{\lambda_k}} \left(\frac{\lambda_{k+1}}{\lambda_k}\right)^{4(\alpha-1)/3} \left[\left(1-\frac{1}{\lambda_k}\right)^{\lambda_k}\right]^{\frac{\lambda_{k+1}}{\lambda_k}} \tag{5.39}$$

for $-\pi \le \theta \le \pi$ and for all large k. Since $r_k^{\lambda_k} \to \frac{1}{e}$ and $n^\gamma y^n \to 0$ as $n \to \infty$ where $\gamma > 0$ and $0 < y < 1$, (5.39) implies that $\frac{R_k}{Q_k} \to 0$ uniformly on $[-\pi, \pi]$. Hence

$$\left|R_k(r_k e^{i\theta})\right| \le \frac{1}{4} \left|Q_k(r_k e^{i\theta})\right| \tag{5.40}$$

for $-\pi \le \theta \le \pi$ and for all large k.

The relations (5.35), (5.38) and (5.40) yield $\left|f(r_k e^{i\theta})\right| \ge \frac{1}{2}\left|Q_k(r_k e^{i\theta})\right|$ for $-\pi \le \theta \le \pi$ and for all large k. Hence

$$\left|f(r_k e^{i\theta})\right| \ge \frac{\lambda_k^{\alpha-1}}{2k^2}\left(1-\frac{1}{\lambda_k}\right)^{\lambda_k}$$

which yields $\left|f(r_k e^{i\theta})\right| \ge \frac{\lambda_k^{\alpha-1}}{6k^2}$ for $-\pi \le \theta \le \pi$ and for all large k. Relation (5.27) gives

$$\left|f(r_k e^{i\theta})\right| \ge \frac{\lambda_k^{\alpha-1}}{6}\,\varepsilon\left(1-\frac{1}{\lambda_k}\right)$$

for $-\pi \le \theta \le \pi$ and for all large k. Hence (5.25) holds with $A = \frac{1}{6}$. □

If $\varepsilon(r) = (1-r)^{\alpha-1}$ where $\alpha > 1$, then (5.23) yields $\overline{\lim_{r\to 1-}} M_p(r, f) = \infty$ for every $p > 0$. Therefore, for any $\alpha > 1$ there is a function belonging to $\mathcal{F}_\alpha$ which belongs to no H^p space. Moreover, (5.22) implies that if $\alpha > 1$ then there is a function $f \in \mathcal{F}_\alpha$ such that $\overline{\lim_{r\to 1-}} \left| f(re^{i\theta}) \right| = \infty$ for all θ in [–π, π] and hence f has no radial limits.

The next theorem shows that Theorem 5.1 is sharp in a strong sense.

Theorem 5.6 *Suppose that* $\alpha > 1$ *and* ε *is a positive function on* (0, 1) *such that* $\lim_{r\to 1-} \varepsilon(r) = 0$ *and* $\lim_{r\to 1-} \frac{\varepsilon(r)}{(1-r)^{\alpha-1}} = \infty$. *There is a function* $f \in \mathcal{F}_\alpha$ *such that if* $\{z_k\}$ $(k = 1, 2, \ldots)$ *denotes the zeros of* f *in* $\{z: 0 < |z| < 1\}$, *listed by multiplicity, and* $|z_k|$ *is nondecreasing, then*

$$\overline{\lim_{n\to\infty}} \left\{ \frac{1}{n^{\alpha-1}\, \varepsilon\left(1 - \frac{1}{n}\right)} \prod_{k=1}^{n} \frac{1}{|z_k|} \right\} > 0. \tag{5.41}$$

Proof: We use the function f constructed in the proof of Theorem 5.5 where the function ε has the additional property that $\lim_{r\to 1-} \frac{\varepsilon(r)}{(1-r)^{\alpha-1}} = \infty$. Then (5.25) implies

$$\int_{-\pi}^{\pi} \log \left| f(r_k e^{i\theta}) \right| d\theta \geq 2\pi \log \frac{A\, \varepsilon(r_k)}{(1-r_k)^{\alpha-1}}$$

and therefore

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \log \left| f(re^{i\theta}) \right| d\theta = \infty.$$

This implies that f has an infinite number of zeros. Let $\{z_n\}$ $(n = 1, 2, \ldots)$ denote the zeros of f in $\{z: 0 < |z| < 1\}$, listed by multiplicity. Then since $f \neq 0$, $\sum_{n=1}^{\infty} (1 - |z_n|) = \infty$. We may assume that $\{|z_n|\}$ is nondecreasing. For $0 < r < 1$ let $n(r)$ denote the number of terms in the sequence $\{z_n\}$ with $|z_n| < r$. Then the zeros of f in $\{z: 0 < |z| < r\}$ are $z_1, z_2, \ldots, z_{n(r)}$ and Jensen's formula and (5.30) give

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left|f(re^{i\theta})\right|\,d\theta = \sum_{j=1}^{n(r)}\log\frac{r}{|z_j|} + (\alpha-1)\log\lambda_1 + \lambda_1\log r$$

for $0 < r < 1$. In this formula let $r = r_k$ ($k = 2, 3, \ldots$) and let $n_k = n(r_k)$. Then

$$\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left|f(r_k e^{i\theta})\right|\,d\theta\right] = \lambda_1^{\alpha-1}\; r_k^{\lambda_1}\; r_k^{n_k}\prod_{j=1}^{n_k}\frac{1}{|z_j|}. \tag{5.42}$$

Hence (5.25) yields

$$\lambda_1^{\alpha-1}\; r_k^{\lambda_1}\; r_k^{n_k}\prod_{j=1}^{n_k}\frac{1}{|z_j|} \geq \frac{A\,\varepsilon(r_k)}{(1-r_k)^{\alpha-1}} \tag{5.43}$$

for all large k.

Let P_k, Q_k and R_k be defined by (5.32), (5.33) and (5.34). Relations (5.38) and (5.40) imply that $|P_k(r_k e^{i\theta}) + R_k(r_k e^{i\theta})| \leq \frac{1}{2}|Q_k(r_k e^{i\theta})| < |Q_k(r_k e^{i\theta})|$ for $-\pi \leq \theta \leq \pi$ and for all large k. Hence (5.35) yields $|f(z) - Q_k(z)| < |Q_k(z)|$ for $|z| = r_k$. This implies that $f(z) \neq 0$ for $|z| = r_k$ and by Rouché's theorem f and Q_k have the same number of zeros in the disk $\{z\colon\ |z| < r_k\}$. Therefore

$$n_k = \lambda_k \tag{5.44}$$

for all sufficiently large k.

From (5.31), (5.43) and (5.44) we obtain

$$\frac{1}{\lambda_k^{\alpha-1}\,\varepsilon(1-\frac{1}{\lambda_k})}\prod_{j=1}^{\lambda_k}\frac{1}{|z_j|} \geq \frac{A}{\left(1-\frac{1}{\lambda_k}\right)^{\lambda_k}\lambda_1^{\alpha-1}\left(1-\frac{1}{\lambda_k}\right)^{\lambda_1}}.$$

This implies that there is a positive constant B such that

$$\frac{1}{\lambda_k^{\alpha-1}\,\varepsilon(1-\frac{1}{\lambda_k})}\prod_{j=1}^{\lambda_k}\frac{1}{|z_j|} \geq B$$

for all sufficiently large k. This proves (5.41). □

In the proofs of Theorems 5.5 and 5.6 the function g was introduced in (5.29). Since g 0 H^∞ the results actually concern the fractional derivatives of functions in H^∞. In order to emphasize this, we focus on what was shown in the case $\alpha = 2$. The relations (5.29) and (5.30) show that if $\alpha = 2$, then
$f(z) = zg'(z)$. Hence the results concern the growth and the zeros of the derivatives of functions in H^∞. The first statement in the next theorem should be compared with the fact that if f 0 H^∞ then $f'(z) = O\left(\frac{1}{1-|z|}\right)$. The second statement follows from Theorem 5.1, $H^\infty \subset \mathcal{F}_1$ and Theorem 2.8.

Theorem 5.7 (a) *Suppose that* ε *is a positive function on* (0,1) *such that* $\lim_{r\to 1-} \varepsilon(r) = 0$. *Then there exists a function* f 0 H^∞ *such that*

$$\overline{\lim_{r\to 1-}} \frac{(1-r)}{\varepsilon(r)} \min_{|z|=r} |f'(z)| = \infty .$$

(b) *Suppose that* f 0 H^∞, *f is nonconstant and the zeros of* f' *in* $\{z: 0 < |z| < 1\}$ *are given by* $\{z_n\}$ $(n = 1, 2, \ldots)$ *where* $\{|z_n|\}$ *is nondecreasing. Then* $\lim_{n\to\infty} \frac{1}{n} \prod_{k=1}^{n} \frac{1}{|z_k|} = 0$.

(c) *Suppose that* ε *is a positive function on* (0, 1) *such that* $\lim_{r\to 1-} \varepsilon(r) = 0$ *and* $\lim_{r\to 1-} \frac{\varepsilon(r)}{1-r} = \infty$. *Then there is a nonconstant function* f 0 H^∞ *such that* f' *has an infinite number of zeros and if* $\{z_k\}$ $(k = 1, 2, \ldots)$ *are the zeros of* f' in $\{z: 0 < |z| < 1\}$, *listed by multiplicity, and where* $\{|z_k|\}$ *is nondecreasing, then*

$$\overline{\lim_{n\to\infty}} \left\{ \frac{1}{n\,\varepsilon\left(1 - \frac{1}{n}\right)} \prod_{k=1}^{n} \frac{1}{|z_k|} \right\} > 0 .$$

NOTES

Theorems 1, 4, 5 and 6 are due to Hallenbeck and MacGregor [1993a]. Theorem 2 was proved by H.S. Shapiro and Shields [1962]. The main argument for Theorem 4 is a construction of Carleson [1952].

CHAPTER 6

Multipliers: Basic Results

<u>Preamble</u>. Let $\mathcal{F}$ and $\mathcal{G}$ denote two families of complex-valued functions defined on a set S δ |. A complex-valued function f defined on S is called a multiplier of $\mathcal{G}$ into $\mathcal{F}$ if $f \cdot g$ belongs to $\mathcal{F}$ for every g in $\mathcal{G}$. For example, Theorem 2.7 asserts that each function in $\mathcal{F}_\alpha$ is a multiplier of $\mathcal{F}_\beta$ into $\mathcal{F}_{\alpha+\beta}$ for each $\alpha > 0$ and $\beta > 0$.

This chapter begins a study of the functions that multiply $\mathcal{F}_\alpha$ into itself. We generally focus on the case $\alpha > 0$. Further information about these multipliers is obtained in Chapter 7.

Let $\mathcal{M}_\alpha$ denote the set of multipliers of $\mathcal{F}_\alpha$. If f belongs to $\mathcal{M}_\alpha$, then the map $g \mapsto f \cdot g$ is a continuous linear operator on $\mathcal{F}_\alpha$. The norm of this operator gives a norm on $\mathcal{M}_\alpha$ and $\mathcal{M}_\alpha$ is a Banach space with respect to that norm.

Functions in $\mathcal{M}_\alpha$ have a number of properties. We find that if f belongs to $\mathcal{M}_\alpha$, then f is bounded. More generally, certain weighted partial sums of f are uniformly bounded. A multiplier maps each radial line segment in $\mathbb{D}$ onto a rectifiable curve. Also we show that $\mathcal{M}_\alpha$ δ $\mathcal{F}_\alpha$ and if $0 \leq \alpha < \beta$ then $\mathcal{M}_\alpha$ δ $\mathcal{M}_\beta$.

An analytic function belongs to $\mathcal{M}_\alpha$ ($\alpha > 0$) if and only if it multiplies the kernels $z \mapsto \dfrac{1}{(1-\overline{\zeta}z)^\alpha}$ boundedly for $|\zeta| = 1$. Many of the results about $\mathcal{M}_\alpha$ follow from this fundamental fact.

A basic sufficient condition for membership of a function in $\mathcal{M}_\alpha$ is proved for the case $0 < \alpha < 1$. This condition concerns the boundedness of certain weighted radial variations of the function. Several other sufficient conditions are derived in Chapter 7. Some of these subsequent conditions rely on the basic condition given here.

Let $\alpha \geq 0$. Let $\mathcal{M}_\alpha$ denote the set of multipliers of $\mathcal{F}_\alpha$, that is, f 0 $\mathcal{M}_\alpha$ provided that $f \cdot g$ 0 $\mathcal{F}_\alpha$ for every g 0 $\mathcal{F}_\alpha$.

Some facts about $\mathcal{M}_\alpha$ hold in the more general setting of Banach spaces with additional properties described in the next lemma.

Lemma 6.1 *Let $\mathcal{B}$ be a Banach space of complex-valued functions defined on a set* $S \subseteq \mathbb{C}$, *where*

(a) *For each* $s \in S$, *point evaluation at* s *is a bounded linear functional on* $\mathcal{B}$, *and*

(b) *For each* $s \in S$ *there exists* $g \in \mathcal{B}$ *such that* $g(s) \neq 0$.

Let $f : S \to \mathbb{C}$ *have the property that* $f \cdot g \in \mathcal{B}$ *for every* $g \in \mathcal{B}$. *Then the mapping* M_f *defined by*

$$M_f(g) = f \cdot g \tag{6.1}$$

for $g \in \mathcal{B}$ *is a bounded linear operator on* $\mathcal{B}$. *Also the function f is bounded.*

Proof: By assumption M_f is well-defined. Also M_f is linear. By the closed graph theorem, M_f is bounded if $G \equiv \{(g, f \cdot g) : g \in \mathcal{B}\}$ is closed in $\mathcal{B} \times \mathcal{B}$. Let $g_n \to g$ and let $f \cdot g_n \to h$. It suffices to show that $h = f \cdot g$. Let $s \in S$. By assumption (a) above,

$$\begin{aligned} |h(s) - f(s)\,g(s)| &\le |h(s) - f(s)\,g_n(s)| + |(g_n(s) - g(s))\, f(s)| \\ &\le A\, \| h - f \cdot g_n \|_{B} + A\, |f(s)|\, \| g_n - g \|_{B} \end{aligned}$$

where A is a positive constant. Letting $n \to \infty$ yields $h(s) = f(s)\,g(s)$.

For $s \in S$ let λ_s denote evaluation at s, that is, $\lambda_s(g) = g(s)$ for $g \in \mathcal{B}$. Assumption (b) above implies that $\|\lambda_s\| > 0$. For each $g \in \mathcal{B}$,

$$\| f \cdot g \|_{B} \le \| M_f \| \, \| g \|_{B}.$$

Let $g \in \mathcal{B}$ and let $s \in S$. Then

$$| f(s)\, g(s) | \le \| \lambda_s \| \, \| f \cdot g \|_{B} \le \| \lambda_s \| \, \| M_f \| \, \| g \|_{B}.$$

It follows that

$$\sup_{\|g\|_{B}=1} | f(s) | \, | g(s) | \le \| \lambda_s \| \, \| M_f \|$$

and therefore

$$| f(s) | \; \| \lambda_s \| \leq \| \lambda_s \| \, \| M_f \| .$$

Since $\| \lambda_s \| > 0$ this yields $| f(s) | \leq \| M_f \|$.

Theorem 6.2 *Suppose that* $f \; 0 \; \mathcal{M}_\alpha$ *for some* $\alpha \geq 0$ *and define* $M_f : \mathcal{F}_\alpha \to \mathcal{F}_\alpha$ *by* (6.1). *Then* M_f *is a continuous linear operator on* $\mathcal{F}_\alpha$ *and* $f \; 0 \; H^4$.

Proof: It suffices to verify that conditions (a) and (b) in Lemma 6.1 hold for $\mathcal{F}_\alpha$. In Chapter 1 we showed that the mapping $f \mapsto f(z)$ is a bounded linear functional on $\mathcal{F}_\alpha$ for each $z \; 0 \; \mathcal{D}$ and for each $\alpha \geq 0$. To verify (b) note that the function $g(z) = 1 + z$ belongs to $\mathcal{F}_\alpha$ for all $\alpha \geq 0$ and g has no zeros in $\mathcal{D}$.

Suppose that $f \; 0 \; \mathcal{M}_\alpha$ for some $\alpha \geq 0$. The operator norm of M_f on $\mathcal{F}_\alpha$ is denoted by $\| M_f \|_\alpha$ and is defined by

$$\| M_f \|_\alpha = \sup_{\substack{g \in \mathcal{F}_\alpha \\ g \neq 0}} \frac{\| M_f(g) \|_{\mathcal{F}_\alpha}}{\| g \|_{\mathcal{F}_\alpha}} . \tag{6.2}$$

Equivalently

$$\| M_f \|_\alpha = \sup_{\| g \|_{\mathcal{F}_\alpha} \leq 1} \| M_f(g) \|_{\mathcal{F}_\alpha} .$$

We define the multiplier norm of f to be this operator norm, that is,

$$\| f \|_{\mathcal{M}_\alpha} = \| M_f \|_\alpha . \tag{6.3}$$

Then $\mathcal{M}_\alpha$ is a normed vector space with respect to the norm (6.3). By the argument given in the proof of Lemma 6.1, $\| f \|_{H^\infty} \leq \| f \|_{\mathcal{M}_\alpha}$.

Theorem 6.3 *For each* $\alpha \geq 0$, $\mathcal{M}_\alpha \; \delta \; \mathcal{F}_\alpha$ *and* $\| f \|_{\mathcal{F}_\alpha} \leq \| f \|_{\mathcal{M}_\alpha}$.

Proof: Let I denote the function $I(z) = 1$ for $|z| < 1$. In the proof of Lemma 2.9 we noted that $I \; 0 \; \mathcal{F}_\alpha$ for $\alpha > 0$, and $\| I \|_{\mathcal{F}_\alpha} = 1$ since I can be represented using a probability measure in (1.1). In the case $\alpha = 0$ note that if μ is the zero measure then

$$1 = 1 + \int_T \log \frac{1}{1 - \overline{\zeta} z} \, d\mu(\zeta)$$

and hence $\| I \|_{F_0} = 1$. Thus $\| I \|_{F_\alpha} = 1$ for $\alpha \geq 0$.

If $\alpha \geq 0$ and f 0 $\mathcal{M}_\alpha$, then $f = f \cdot I$ and hence

$$\| f \|_{F_\alpha} \leq \| f \|_{M_\alpha} \| I \|_{F_\alpha} = \| f \|_{M_\alpha} .$$

Theorem 6.4 *For each* $\alpha \geq 0$, $\mathcal{M}_\alpha$ *is a Banach space.*

Proof: It suffices to show that $\mathcal{M}_\alpha$ is complete in the norm (6.3). Suppose that $\{ f_n \}$ (n = 1, 2, …) is a Cauchy sequence in $\mathcal{M}_\alpha$, and let $\varepsilon > 0$. There exists a positive integer N such that

$$\| f_n - f_m \|_{M_\alpha} < \varepsilon \tag{6.4}$$

for n, m $\geq$ N. Hence Theorem 6.3 implies that $\{ f_n \}$ is a Cauchy sequence in the norm of $\mathcal{F}_\alpha$. Since $\mathcal{F}_\alpha$ is a Banach space, there exists f 0 $\mathcal{F}_\alpha$ such that $\| f_n - f \|_{F_\alpha} \to 0$ as $n \to 4$. This implies that $f_n \to f$ uniformly on compact subsets of $\mathcal{D}$.

We shall show that f 0 $\mathcal{M}_\alpha$. Suppose that g 0 $\mathcal{F}_\alpha$ and $\| g \|_{F_\alpha} \leq 1$. If m, n $\geq$ N then

$$\| (f_n - f_m) \cdot g \|_{F_\alpha} \leq \| f_n - f_m \|_{M_\alpha} \| g \|_{F_\alpha} \leq \| f_n - f_m \|_{M_\alpha} < \varepsilon .$$

Thus the sequence $\{ f_n \cdot g \}$ (n = 1, 2, …) is Cauchy in the norm of $\mathcal{F}_\alpha$, and it follows that there exists h 0 $\mathcal{F}_\alpha$ with

$$\| f_n \cdot g - h \|_{F_\alpha} \to 0 \tag{6.5}$$

as $n \to 4$. This implies that $f_n \cdot g \to h$ uniformly on compact subsets of $\mathcal{D}$. For each z 0 $\mathcal{D}$ we have $f_n(z) \to f(z)$ and $f_n(z)\, g(z) \to h(z)$. Therefore

$$f \cdot g = h . \tag{6.6}$$

Since h 0 $\mathcal{F}_\alpha$, this proves that f 0 $\mathcal{M}_\alpha$.

The relations (6.5) and (6.6) show that $\| f_n \cdot g - f \cdot g \|_{F_\alpha} \to 0$. This holds for every g 0 $\mathcal{F}_\alpha$ with $\| g \|_{F_\alpha} \leq 1$. Therefore $\| f_n - f \|_{M_\alpha} \to 0$. This proves that $\mathcal{M}_\alpha$ is complete.

Theorem 6.5 *Let $\alpha > 0$ and let the function f be analytic in $\mathcal{D}$. The following are equivalent.*

(a) $f \; 0 \; \mathcal{M}_\alpha$.

(b) *There is a positive constant* M *such that*

$$\| f(z) \cdot \frac{1}{(1-\overline{\zeta}z)^\alpha} \|_{F_\alpha} \leq M \tag{6.7}$$

for all $|\zeta| = 1$.

Proof: First assume that $f \; 0 \; \mathcal{M}_\alpha$. Theorem 6.2 implies that

$$M \equiv \sup \{ \| M_f(g) \|_{F_\alpha} : g \; 0 \; F_\alpha \text{ and } \| g \|_{F_\alpha} \leq 1 \} < \infty$$

and $\| f \cdot g \|_{F_\alpha} \leq M \| g \|_{F_\alpha}$ for all $g \; 0 \; F_\alpha$. Let $g(z) = \dfrac{1}{(1-\overline{\zeta}z)^\alpha}$ where $|\zeta| = 1$. Then $\| g \|_{F_\alpha} = 1$ and condition (b) is established.

For the converse, assume that condition (b) holds, and let $g \; 0 \; F_\alpha$. Then

$$g(z) = \int_T \frac{1}{(1-\overline{\zeta}z)^\alpha} \, d\mu(\zeta) \qquad (|z| < 1) \tag{6.8}$$

for some $\mu \; 0 \; \mathcal{M}$. To show that $f \cdot g \; 0 \; F_\alpha$, we may assume that $\mu \; 0 \; \mathcal{M}^*$. Then g is the limit of a sequence of functions each of which has the form

$$h(z) = \sum_{k=1}^{n} b_k \frac{1}{(1-\overline{\zeta}_k z)^\alpha} \tag{6.9}$$

where $b_k \geq 0$, $\sum_{k=1}^{n} b_k = 1$, $|\zeta_k| = 1$ and n is some positive integer, and the limit is uniform on compact subsets of $\mathcal{D}$.

Suppose that h is defined by (6.9) with the stated conditions on b_k, ζ_k and n. For k = 1, 2, …, n there exists $\upsilon_k \; 0 \; \mathcal{M}$ such that

$$f(z) \cdot \frac{1}{(1-\overline{\zeta}_k z)^{\alpha}} = \int_T \frac{1}{(1-\overline{\zeta} z)^{\alpha}}\, d\upsilon_k(\zeta) \qquad (|z| < 1) \tag{6.10}$$

and $\| \upsilon_k \| \le M$. Let $\upsilon = \sum_{k=1}^{n} b_k \upsilon_k$. Then $\upsilon \, 0 \, \mathcal{M}$ and (6.9) and (6.10) imply that

$$f(z) \cdot h(z) = \int_T \frac{1}{(1-\overline{\zeta} z)^{\alpha}}\, d\upsilon(\zeta) \tag{6.11}$$

for $|z| < 1$. Also $\| \upsilon \| \le \sum_{k=1}^{n} b_k \, \| \upsilon_k \| \le M \sum_{k=1}^{n} b_k = M$.

By the Banach-Alaoglu theorem, the set $\{ \upsilon \;\, 0 \; \mathcal{M} : \| \upsilon \| \le M \}$ is compact in the weak* topology. Hence an argument using subsequences shows that there exists $\lambda \; 0 \; \mathcal{M}$ such that $\|\lambda\| \le M$ and

$$f(z) \cdot g(z) = \int_T \frac{1}{(1-\overline{\zeta} z)^{\alpha}}\, d\lambda(\zeta)$$

for $|z| < 1$. Hence $f \cdot g \; 0 \; \mathcal{F}_{\alpha}$ and this establishes condition (a).

Theorem 6.6 *If* $0 < \alpha < \beta$ *then* $\mathcal{M}_{\alpha} \;\delta\; \mathcal{M}_{\beta}$.

Proof: Let $f \; 0 \; \mathcal{M}_{\alpha}$ where $0 < \alpha < \beta$. Theorem 6.5 implies that there is a positive constant M such that

$$\left\| f(z) \cdot \frac{1}{(1-\overline{\zeta} z)^{\alpha}} \right\|_{\mathcal{F}_{\alpha}} \le M \tag{6.12}$$

for all $|\zeta| = 1$. For all such ζ, the function $\dfrac{1}{(1-\overline{\zeta} z)^{\beta-\alpha}}$ belongs to $\mathcal{F}_{\beta-\alpha}$ and $\left\| \dfrac{1}{(1-\overline{\zeta} z)^{\beta-\alpha}} \right\|_{\mathcal{F}_{\beta-\alpha}} = 1$. Theorem 2.7 implies that

$$\| (f(z) \cdot \frac{1}{(1-\overline{\zeta}z)^{\alpha}}) \cdot \frac{1}{(1-\overline{\zeta}z)^{\beta-\alpha}} \|_{F_\beta} \leq \| f(z) \cdot \frac{1}{(1-\overline{\zeta}z)^{\alpha}} \|_{F_\alpha} \ \| \frac{1}{(1-\overline{\zeta}z)^{\beta-\alpha}} \|_{F_{\beta-\alpha}}$$
$$\leq M \cdot 1 = M$$

for all $|\zeta| = 1$. Hence Theorem 6.5 yields $f \; 0 \; \mathcal{M}_\beta$.

Later we discuss the question of whether $\mathcal{M}_\alpha \;\; \mathcal{M}_\beta$ when 0 <_ $\alpha < \beta$.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. For $\alpha > 0$ and $n = 0, 1, 2, \ldots$ let

$$P_n(z, \alpha) = \frac{1}{A_n(\alpha)} \sum_{k=0}^{n} A_{n-k}(\alpha)\, a_k z^k . \tag{6.13}$$

In particular, $\{P_n(z, 1)\}$ $(n = 0, 1, \ldots)$ is the sequence of partial sums of the Taylor series for f.

Theorem 6.7 *If* $f \; 0 \; \mathcal{M}_\alpha$ *for some* $\alpha > 0$ *then*

$$\| P_n(\cdot, \alpha) \|_{H^\infty} \leq \| f \|_{\mathcal{M}_\alpha} \tag{6.14}$$

for $n = 0, 1, 2, \ldots$.

Proof: Suppose that $\alpha > 0$ and $f \; 0 \; \mathcal{M}_\alpha$. Let M be any real number such that $\| f \|_{\mathcal{M}_\alpha} < M$. Let $|\zeta| = 1$. By Theorem 6.5 there exists $\mu_\zeta \; 0 \; \mathcal{M}$ such that

$$f(z) \cdot \frac{1}{(1-\overline{\zeta}z)^{\alpha}} = \int_T \frac{1}{(1-\overline{w}z)^{\alpha}} \, d\mu_\zeta(w) \tag{6.15}$$

for $|z| < 1$ and $\| \mu_\zeta \| \leq M$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and let

$$f(z) \cdot \frac{1}{(1-\overline{\zeta}z)^{\alpha}} = \sum_{n=0}^{\infty} b_n(\zeta)\, z^n$$

for $|z| < 1$. Then

$$b_n(\zeta) = \sum_{k=0}^{n} A_k(\alpha)\, a_{n-k}\, \overline{\zeta}^k \tag{6.16}$$

for n = 0, 1, … . Also

$$\int_T \frac{1}{(1-\overline{w}z)^{\alpha}}\, d\mu_\zeta(w) = \sum_{n=0}^{\infty} A_n(\alpha) \int_T \overline{w}^n\, d\mu_\zeta(w)\, z^n .$$

Hence (6.15) and (6.16) yield

$$\overline{\zeta}^n P_n(\zeta, \alpha) = \int_T \overline{w}^n\, d\mu_\zeta(w) \tag{6.17}$$

for n = 0, 1, … . Because $\| \mu_\zeta \| \leq M$, (6.17) implies that $| P_n(\zeta, \alpha) | \leq M$ for $| \zeta | = 1$ and n = 0, 1, … . This inequality holds for every $M > \| f \|_{\mathcal{M}_\alpha}$, which yields (6.14).

In general, Theorem 6.7 gives a stronger statement than the inclusion $\mathcal{M}_\alpha$ δ H^4 from Theorem 6.2. For example, when α = 1 Theorem 6.7 shows that the partial sums of a function in $\mathcal{M}_1$ are uniformly bounded.

Given a power series $\sum_{n=0}^{\infty} a_n z^n$, let $s_n(z) = \sum_{k=0}^{n} a_k z^k$ and let $\sigma_n(z) = \frac{1}{n+1} \sum_{k=0}^{n} s_k(z)$ for n = 0, 1, … . Note that $\sigma_n(z) = P_n(z, 2)$. It is a classical result that if f 0 H^4 and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\| \sigma_n \|_{H^\infty} \leq \| f \|_{H^\infty}$ for n = 0, 1, … . Conversely, if there is a positive constant M with $\| \sigma_n \|_{H^\infty} \leq M$ for n = 0, 1, …, then f is bounded and $\| f \|_{H^\infty} \leq M$. Similar results hold for the polynomials $P_n(\cdot, \alpha)$ for $\alpha \neq 2$, as stated in the next theorem (a reference is given in the Notes).

Theorem 6.8 *Suppose that f is analytic in* $\mathbb{D}$ *and* $\alpha > 0$.

(a) *If there is a positive constant* M *with* $\| P_n(\cdot, \alpha) \|_{H^\infty} \leq M$ *for* n = 0, 1, … , *then* f 0 H^4.

(b) *For each* $\alpha > 1$ *there is a positive constant* $B(\alpha)$ *such that if* f 0 H^4 *then* $\| P_n(\cdot, \alpha) \|_{H^\infty} \le B(\alpha) \| f \|_{H^\infty}$ for $n = 0, 1, \ldots$. *When* $\alpha \geq 2$ *this inequality holds with* $B(\alpha) = 1$.

Theorem 6.9 *If* f 0 $\mathcal{M}_\alpha$ *for some* $\alpha > 0$ *then* f *has a nontangential limit at every point of* T.

Proof: Let ζ_0 0 T. The hypotheses imply that there is a measure μ 0 $\mathcal{M}$ such that

$$f(z) \cdot \frac{1}{(1-\overline{\zeta}_0 z)^\alpha} = \int_T \frac{1}{(1-\overline{\zeta} z)^\alpha} \, d\mu(\zeta)$$

for $|z| < 1$ and hence

$$f(z) = \int_T \left(\frac{1-\overline{\zeta}_0 z}{1-\overline{\zeta} z} \right)^\alpha d\mu(\zeta). \tag{6.18}$$

The argument now proceeds as in the proof of Theorem 4.10.

Theorem 6.10 *Suppose that* f 0 $\mathcal{M}_\alpha$ *for some* $\alpha > 0$. *For* ζ 0 T, *let* $V(\zeta)$ *denote the radial variation of* f *in the direction* ζ. *Then there is a positive constant* A *depending only on* α *such that* $V(\zeta) \le A \| f \|_{\mathcal{M}_\alpha}$ *for all* $|\zeta| = 1$.

Proof: Suppose that $\alpha > 0$ and f 0 $\mathcal{M}_\alpha$. Let $M > \| f \|_{\mathcal{M}_\alpha}$. For each ζ with $|\zeta| = 1$ there exists μ_ζ 0 $\mathcal{M}$ such that

$$f(z) \cdot \frac{1}{(1-\overline{\zeta} z)^\alpha} = \int_T \frac{1}{(1-\overline{w} z)^\alpha} \, d\mu_\zeta(w)$$

for $|z| < 1$. Also $\| \mu_\zeta \| \le M$ for all such ζ. Hence

$$f(z) = \int_{T \setminus \{\zeta\}} \left(\frac{1-\overline{\zeta} z}{1-\overline{w} z} \right)^\alpha d\mu_\zeta(w) + \mu_\zeta(\{\zeta\})$$

and

$$f'(z) = \alpha \int_{T\backslash\{\zeta\}} \frac{(1-\overline{\zeta}z)^{\alpha-1}\ (\overline{w}-\overline{\zeta})}{(1-\overline{w}z)^{\alpha+1}}\, d\mu_\zeta(w).$$

Since $V(\zeta) = \int_0^1 |\, f'(r\zeta)\,|\, dr$ this yields

$$V(\zeta) \le \alpha \int_{T\backslash\{\zeta\}} \left\{ \int_0^1 \frac{(1-r)^{\alpha-1}\ |\, w-\zeta\,|}{|\,1-r\overline{w}\zeta\,|^{\alpha+1}}\, dr \right\} d\,|\,\mu_\zeta\,|(w). \tag{6.19}$$

Let I denote the inner integral in (6.19). Since

$$|\,1-r\overline{w}\zeta\,|^{\alpha+1} = \left\{(1-r)^2 + r\,|\,1-\overline{w}\zeta\,|^2\right\}^{\frac{\alpha+1}{2}} \ge \left\{(1-r)^2 + r^2\ |\,1-\overline{w}\zeta\,|^2\right\}^{\frac{\alpha+1}{2}}$$

it follows that

$$I \le \int_0^1 \frac{(1-r)^{\alpha-1} b}{\left\{(1-r)^2 + r^2\ b^2\right\}^{(\alpha+1)/2}}\, dr \equiv J,$$

where $b = |\, w-\zeta\,|$. The change of variables $s = \frac{rb}{1-r}$ yields

$$J = \int_0^\infty \frac{1}{(1+s^2)^{(\alpha+1)/2}}\, ds \equiv B_\alpha < \infty.$$

Therefore $V(\zeta) \le \alpha\ B_\alpha \int_{T\backslash\{\zeta\}} d\,|\,\mu_\zeta\,|(w) \le \alpha\ B_\alpha\ \|\,\mu_\zeta\,\| \le \alpha\ B_\alpha\ M$. Since M is any constant with $M > \|\, f\,\|_{M_\alpha}$, the proof is complete.

In the case $\alpha > 0$, Theorem 6.10 yields another proof that $\mathcal{M}_\alpha\ \delta\ H^4$. This follows from the inequality

$$|\, f(z)| \le |\, f(0)| + \int_0^1 |\, f'(r\zeta)\,|\, dr$$

where $z = \rho\zeta$, $0 \leq \rho < 1$ and $|\zeta| = 1$. Theorem 6.10 also implies Theorem 6.9. This is a consequence of the fact that if $V(\zeta) < 4$ then $f(r\zeta)$ is uniformly continuous in r on the interval [0,1). Hence $f(r\zeta)$ extends continuously to r = 1, that is, f has a radial limit in the direction ζ. Because f 0 H^4, Lemma 4.4 shows that f has a nontangential limit at ζ.

Suppose that f is analytic in $\mathcal{D}$ and f extends continuously to $\overline{D}$. Theorem 6.10 shows that this is not sufficient to imply that f 0 $\mathcal{M}_\alpha$ for some $\alpha > 0$. To see this let Ω be a Jordan domain for which there exists w_0 0 $\partial\Omega$ with the property that each continuous curve $w = w(t)$ $(0 \leq t \leq 1)$, where $w(t)$ 0 Ω for $0 \leq t < 1$ and $w(1) = w_0$, has infinite length. Let f be a conformal mapping of $\mathcal{D}$ onto Ω. A theorem of Carathéodory implies that f can be extended to a homeomorphism of $\overline{D}$ onto $\overline{\Omega}$. Let z_0 be the unique point with $|z_0| = 1$ and $f(z_0) = w_0$. Then the image of the line segment from 0 to z_0 is not rectifiable and Theorem 6.10 shows that f ⌠ $\mathcal{M}_\alpha$ for all $\alpha > 0$. In the remarks after Theorem 7.25 we give examples of analytic functions which show that a continuous extension to $\overline{D}$ is not necessary for membership in $\mathcal{M}_\alpha$.

We shall obtain various conditions sufficient to imply that a function analytic in $\mathcal{D}$ belongs to $\mathcal{M}_\alpha$. The condition given in this chapter applies for $0 < \alpha < 1$ and it implies that the function extends continuously to $\overline{D}$ and satisfies a Lipschitz condition of order $1-\alpha$. First we prove this implication. To do so, we need the following two lemmas.

Lemma 6.11 *Suppose that f is analytic in $\overline{D}$ and $0 < \beta < 1$. Let*

$$A = \sup_{\substack{0 \leq r < 1 \\ |\theta| \leq \pi}} \frac{|f(re^{i\theta}) - f(e^{i\theta})|}{(1-r)^\beta} \tag{6.20}$$

and

$$B = \sup_{\substack{h > 0 \\ |\theta| \leq \pi}} \frac{|f(e^{i(\theta+h)}) - f(e^{i\theta})|}{h^\beta}. \tag{6.21}$$

Then $A < 4$, $B < 4$ *and there is a positive constant* C *depending only on* β *such that*

$$B \leq C\,A. \tag{6.22}$$

Proof: Let $0 < \beta < 1$ and suppose that f is analytic in $\overline{D}$. It follows that

$A < 4$ and $B < 4$. Let $z = re^{i\theta}$ where $0 \le r < 1$ and θ is real. The Poisson formula gives

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - \varphi) \; f(e^{i\varphi}) \; d\varphi \tag{6.23}$$

where

$$P(r, \theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2}. \tag{6.24}$$

Differentiation of (6.23) with respect to θ yields

$$\begin{aligned} i z f'(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ -\frac{2r(1-r^2) \sin(\theta - \varphi)}{[1 - 2r \cos(\theta - \varphi) + r^2]^2} \right\} f(e^{i\varphi}) \, d\varphi \\ &= \frac{r(1-r^2)}{\pi} \int_{-\pi}^{\pi} \frac{\sin \omega}{[1 - 2r \cos \omega + r^2]^2} \; f(e^{i(\omega+\theta)}) \, d\omega. \end{aligned}$$

Since

$$\int_{-\pi}^{0} \frac{\sin \omega}{[1 - 2r \cos \omega + r^2]^2} \, f(e^{i(\omega+\theta)}) \, d\omega = -\int_{0}^{\pi} \frac{\sin \sigma}{[1 - 2r \cos \sigma + r^2]^2} \; f(e^{i(\theta-\sigma)}) \, d\sigma$$

it follows that

$$i z f'(z) = \frac{r(1-r^2)}{\pi} \int_{0}^{\pi} \frac{\sin \omega}{[1 - 2r \cos \omega + r^2]^2} \left\{ f(e^{i(\theta+\omega)}) - f(e^{i(\theta-\omega)}) \right\} d\omega.$$

By a simple argument, there is a positive constant D such that

$$\frac{1}{1 - 2r \cos \omega + r^2} \le \frac{D}{(1-r)^2 + \omega^2}$$

for $0 \le r < 1$ and $0 \le \omega \le \pi$. Therefore

$$| f'(z) | \le \frac{1-r^2}{\pi} D^2 \int_0^{\pi} \frac{\sin \omega}{[(1-r)^2 + \omega^2]^2} \left| f(e^{i(\theta+\omega)}) - f(e^{i(\theta-\omega)}) \right| d\omega .$$

Since $\sin \omega \le \omega$ for $0 \le \omega \le \pi$ and

$$| f(e^{i(\theta+\omega)}) - f(e^{i(\theta-\omega)}) | \le B(2\omega)^{\beta}$$

it follows that

$$| f'(z) | \le \frac{B D^2 \; 2^{\beta+1}}{\pi} (1-r) \int_0^{\pi} \frac{\omega^{\beta+1}}{[(1-r)^2 + \omega^2]^2} d\omega . \tag{6.25}$$

The change of variables $\omega = (1-r)\,x$ yields

$$\begin{aligned} \int_0^{\pi} \frac{\omega^{\beta+1}}{[(1-r)^2 + \omega^2]^2} d\omega &= (1-r)^{\beta-2} \int_0^{\pi/(1-r)} \frac{x^{\beta+1}}{[1 + x^2]^2} dx \\ &\le (1-r)^{\beta-2} \int_0^{\infty} \frac{x^{\beta+1}}{[1 + x^2]^2} dx \\ &\equiv (1-r)^{\beta-2} E < \infty , \end{aligned}$$

where the constant E depends only on β. Hence (6.25) implies that there is a positive constant F depending only on β such that

$$| f'(z) | \le \frac{F B}{(1-r)^{1-\beta}} . \tag{6.26}$$

By integrating along an arc of the circle $\{w: |w| = r\}$, (6.26) yields

$$| f(re^{i(\theta+h)}) - f(re^{i\theta}) | \le \frac{F B}{(1-r)^{1-\beta}} r h .$$

Hence (6.20) gives

$$
\begin{aligned}
&| f(e^{i(\theta+h)}) - f(e^{i\theta})| \\
&\le | f(e^{i(\theta+h)}) - f(re^{i(\theta+h)})| + | f(re^{i(\theta+h)}) - f(re^{i\theta})| + | f(re^{i\theta}) - f(e^{i\theta})| \\
&\le A(1-r)^{\beta} + \frac{FBh}{(1-r)^{1-\beta}} + A(1-r)^{\beta}.
\end{aligned}
$$

If we set $s = 1-r$ then $0 < s \le 1$ and the inequality above is equivalent to

$$| f(e^{i(\theta+h)}) - f(e^{i\theta})| \le 2As^{\beta} + \frac{FBh}{s^{1-\beta}}. \tag{6.27}$$

The inequality (6.27) holds for $0 < s \le 1$. By relation (6.20),

$$| f(e^{i(\theta+h)}) - f(e^{i\theta})| \le 2A$$

and thus (6.27) is valid for $s > 1$. Therefore (6.27) holds for all θ, h and s where θ is real, $h > 0$ and $s > 0$.

The relation (6.21) implies that there exist θ and h with $|\theta| \le \pi$, $h > 0$ and

$$\frac{| f(e^{i(\theta+h)}) - f(e^{i\theta})|}{h^{\beta}} > \frac{1}{2} B.$$

For such θ and h, (6.27) yields

$$\tfrac{1}{2} B h^{\beta} < 2As^{\beta} + \frac{FBh}{s^{1-\beta}}$$

for all $s > 0$. Therefore

$$\tfrac{1}{2} B < 2A \left(\frac{s}{h}\right)^{\beta} + FB\left(\frac{h}{s}\right)^{1-\beta}. \tag{6.28}$$

for a fixed $h > 0$ and for all $s > 0$. The substitution $s = h\,(4F)^{1/(1-\beta)}$ yields

$$\tfrac{1}{2} B < 2A\,(4F)^{\beta/(1-\beta)} + \tfrac{1}{4} B.$$

Therefore $B < 8\,(4F)^{\beta/(1-\beta)} A$. This proves (6.22) for a constant C depending only on β. □

Lemma 6.12 *Suppose that* f 0 H^1 *and there are positive constants* J *and* β *and a set* E δ [−π, π] *having Lebesgue measure* 2π *such that*

$$| f(re^{i\theta}) - f(e^{i\theta}) | \le J(1-r)^{\beta} \tag{6.29}$$

for θ 0 E and $0 \le r < 1$. Then

$$| f(\rho re^{i\theta}) - f(\rho e^{i\theta}) | \le J(1-r)^{\beta} \tag{6.30}$$

for $0 \le \rho < 1$, $0 \le r < 1$ and θ real.

Proof: Suppose that f 0 H^1. An application of the Poisson formula to f and to the function z |→ f (rz) ($0 \le r < 1$) yields

$$f(\rho re^{i\theta}) - f(\rho e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-\rho^2}{1-2\rho\cos(\theta - t) + \rho^2} \left\{ f(re^{it}) - f(e^{it}) \right\} dt$$

for $0 \le \rho < 1$ and θ real. The assumption (6.29) implies that

$$\begin{aligned} | f(\rho re^{i\theta}) - f(\rho e^{i\theta}) | &\le \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-\rho^2}{1-2\rho\cos(\theta - t) + \rho^2} J(1-r)^{\beta} \, dt \\ &= J(1-r)^{\beta}. \end{aligned}$$

Theorem 6.13 *Suppose that the function* f *is analytic in* $\mathbb{D}$ *and*

$$I = I_{\alpha}(f) \equiv \sup_{|\zeta|=1} \int_0^1 | f'(r\zeta) | (1-r)^{\alpha-1} \, dr < \infty \tag{6.31}$$

for some α *with* $0 < \alpha < 1$. *Then* f *extends continuously to* $\overline{D}$ *and* f *satisfies the Lipschitz condition*

$$| f(e^{i(\theta+h)}) - f(e^{i\theta}) | \le C I h^{1-\alpha} \tag{6.32}$$

for $h > 0$ *and* θ *real, where* C *is a positive constant depending only on* α.

Proof: Since $0 < \alpha < 1$, the assumption (6.31) implies that

$$\sup_{|\zeta|=1} \int_0^1 | f'(r\zeta)| \, dr < \infty$$

and hence $f(e^{i\theta}) \equiv \lim_{r\to 1-} f(re^{i\theta})$ exists for all θ. Let Ψ denote the line segment from $re^{i\theta}$ to $e^{i\theta}$. Then $f(e^{i\theta}) - f(re^{i\theta}) = \int_\psi f'(w)\, dw$ and

$$| f(e^{i\theta}) - f(re^{i\theta})| \le \int_r^1 | f'(te^{i\theta})| \, dt \le (1-r)^{1-\alpha} \int_0^1 | f'(te^{i\theta})| \, (1-t)^{\alpha-1} \, dt.$$

Therefore (6.31) yields

$$| f(e^{i\theta}) - f(re^{i\theta})| \le I(1-r)^{1-\alpha} \tag{6.33}$$

for $0 \le r < 1$ and $|\theta| \le \pi$. Lemma 6.12 implies

$$| f(\rho e^{i\theta}) - f(\rho re^{i\theta})| \le I(1-r)^{1-\alpha} \tag{6.34}$$

for $0 \le r < 1$, $0 \le \rho < 1$ and $|\theta| \le \pi$.

Define the function $F = F_\rho$ by $F(z) = f(\rho z)$ where ρ is fixed, $0 < \rho < 1$ and $|z| < 1/\rho$. Then F is analytic in $\overline{D}$. Let

$$A' = \sup \left\{ \frac{|F(re^{i\theta}) - F(e^{i\theta})|}{(1-r)^{1-\alpha}} : 0 \le r < 1, |\theta| \le \pi \right\}$$

and let

$$B' = \sup \left\{ \frac{|F(e^{i(\theta+h)}) - F(e^{i\theta})|}{h^{1-\alpha}} : h > 0, |\theta| \le \pi \right\}.$$

By Lemma 6.11 there is a constant C depending only on α such that $B' \le C\, A'$. The relation (6.34) implies that $A' \le I$ and hence $B' \le C\, I$, that is,

$$| f(\rho e^{i(\theta+h)}) - f(\rho e^{i\theta})| \le C I h^{1-\alpha}$$

for all ρ, $0 \le \rho < 1$. Letting $\rho \to 1-$ in this inequality yields

$$|f(e^{i(\theta+h)}) - f(e^{i\theta})| \le C\,I\,h^{1-\alpha}.$$

This proves (6.32).

To show that f is continuous at points on T, let $|\theta| \le \pi$ and let $z = re^{i\varphi}$ where $0 \le r < 1$ and $|\varphi| \le \pi$. Let $h = |\varphi-\theta|$. Then

$$|f(z) - f(e^{i\theta})| \le |f(re^{i\varphi}) - f(e^{i\varphi})| + |f(e^{i\varphi}) - f(e^{i\theta})|$$

and (6.32) yields

$$|f(z) - f(e^{i\theta})| \le |f(re^{i\varphi}) - f(e^{i\varphi})| + C\,I\,h^{1-\alpha}$$

The function $r \mapsto f(re^{i\theta})$, where $0 \le r < 1$, extends continuously at $r = 1$. Thus the previous inequality shows that $|f(z) - f(e^{i\theta})| \to 0$ as $r \to 1-$ and $h \to 0$.

In order to prove the main sufficient condition for membership in $\mathcal{M}_\alpha$ when $0 < \alpha < 1$, we need the following technical lemmas.

Lemma 6.14 *For each* $\alpha > 0$ *there is a positive constant* A *such that*

$$\int_0^r \frac{1}{|1-\rho e^{i\varphi}|^{\alpha+1}}\, d\rho \le \frac{A}{|1-re^{i\varphi}|^{\alpha}} \qquad (6.35)$$

for $0 \le r < 1$ *and* $-\pi \le \varphi \le \pi$.

Proof: Suppose that $0 \le \rho < 1$ and $-\pi \le \varphi \le \pi$. Then

$$|1-\rho e^{i\varphi}|^2 = (1-\rho)^2 + 2\rho\,(1-\cos\varphi).$$

Since $1 - \cos\varphi \le \varphi^2/2$ this yields

$$\frac{1}{(1-\rho)^2 + \varphi^2} \le \frac{1}{|1-\rho e^{i\varphi}|^2}. \qquad (6.36)$$

There is a constant δ such that $0 < \delta < \pi/2$ and $1 - \cos\varphi \ge \varphi^2/3$ for $|\varphi| \le \delta$. Hence, if $\rho \ge ½$ and $|\varphi| \le \delta$ then

$$\frac{1}{|1-\rho e^{i\varphi}|^2} \le \frac{3}{(1-\rho)^2 + \varphi^2}.$$

If $|z| < 1$, $z = \rho e^{i\varphi}$ and $z \notin \{\rho e^{i\varphi}\colon\ \rho \geq ½ \text{ and } |\varphi| \leq \delta\}$, then $(1-\rho)^2 + \delta^2 \leq A\,|1-\rho e^{i\varphi}|^2$ for a positive constant A. Therefore there is a positive constant B such that

$$\frac{1}{|1-\rho e^{i\varphi}|^2} \leq \frac{B}{(1-\rho)^2 + \varphi^2} \tag{6.37}$$

for $0 \leq \rho < 1$ and $-\pi \leq \varphi \leq \pi$.

For $\alpha > 0$, $0 \leq r < 1$ and $-\pi \leq \varphi \leq \pi$ let

$$I_\alpha = I_\alpha(r,\varphi) = \int_0^r \frac{1}{|1-\rho e^{i\varphi}|^{\alpha+1}}\, d\rho .$$

Suppose that $\varphi > 0$. Then (6.37) and the change of variable $1-\rho = \varphi x$ yield

$$\begin{aligned} I_\alpha &\leq B^{(\alpha+1)/2} \int_0^r \frac{1}{[(1-\rho)^2 + \varphi^2]^{(\alpha+1)/2}}\, d\rho \\ &= \frac{B^{(\alpha+1)/2}}{\varphi^\alpha} \int_{(1-r)/\varphi}^{1/\varphi} \frac{1}{(1+x^2)^{(\alpha+1)/2}}\, dx \\ &\leq \frac{B^{(\alpha+1)/2}}{\varphi^\alpha} \int_{(1-r)/\varphi}^{\infty} \frac{1}{(1+x^2)^{(\alpha+1)/2}}\, dx \\ &= \frac{B^{(\alpha+1)/2}}{\varphi^\alpha}\, J\big((1-r)/\varphi\big) \end{aligned}$$

where

$$J(y) = \int_y^\infty \frac{1}{(1+x^2)^{(\alpha+1)/2}}\, dx \tag{6.38}$$

for $y \geq 0$. If $0 \leq y \leq 1$ then

$$J(y) \leq J(0) \leq \left(\frac{2}{1+y^2}\right)^{\alpha/2} J(0).$$

On the other hand, if y > 1 then

$$J(y) \le \int_y^\infty \frac{1}{x^{\alpha+1}}\,dx = \frac{1}{\alpha y^\alpha}.$$

There is a positive constant C depending only on α such that

$$\frac{1}{\alpha y^\alpha} \le \frac{C}{(1+y^2)^{\alpha/2}}$$

for y > 1. Thus there is a positive constant D depending only on α such that

$$J(y) \le \frac{D}{(1+y^2)^{\alpha/2}} \tag{6.39}$$

for $0 \le y < \infty$. Therefore

$$I_\alpha(r,\varphi) \le \frac{B^{(\alpha+1)/2}\,D}{\varphi^\alpha\{1+[(1-r)/\varphi]^2\}^{\alpha/2}} = \frac{B^{(\alpha+1)/2}\,D}{[(1-r)^2+\varphi^2]^{\alpha/2}}$$

for $0 \le r < 1$ and $0 < \varphi \le \pi$. Since $I_\alpha(r,\varphi) = I_\alpha(r,-\varphi)$, we have

$$I_\alpha \le \frac{E}{[(1-r)^2+\varphi^2]^{\alpha/2}}$$

for $0 \le r < 1$ and $0 < |\varphi| \le \pi$, where E is a positive constant depending only on α. Hence (6.36) with $\rho = r$ yields (6.35) for $0 \le r < 1$ and $0 < |\varphi| \le \pi$. Since $I_\alpha(r,0) \le \frac{1}{\alpha}(1-r)^{-\alpha}$, this completes the proof of (6.35).

Lemma 6.15 *For each* α, 0 < α < 1, *there is a positive constant* D *depending only on* α *such that*

$$\int_0^1\int_0^\pi \frac{\varphi^{1-\alpha}}{|1-\rho e^{i\varphi}|^{\alpha+1}}(1-\rho)^{\alpha-1}\,d\varphi\,d\rho \le D. \tag{6.40}$$

Proof: Fix α with 0 < α < 1. For $0 \le \rho < 1$ let

$$I(\rho) = \int_0^{\pi} \frac{\varphi^{1-\alpha}}{|1-\rho e^{i\varphi}|^{\alpha+1}} \, d\varphi$$

and let

$$J = \int_0^1 I(\rho)\,(1-\rho)^{\alpha-1}\, d\rho .$$

The inequality (6.37) and the change of variable $\varphi = (1-\rho)x$ yield

$$I(\rho) \le B^{(\alpha+1)/2}\,(1-\rho)^{1-2\alpha} \int_0^{\pi/(1-\rho)} \frac{x^{1-\alpha}}{(1+x^2)^{(\alpha+1)/2}}\, dx .$$

There is a constant E depending only on α such that $x^{1-\alpha}/(1+x^2)^{(\alpha+1)/2} \le E\, x^{-2\alpha}$ for $x \ge \pi$. Hence if $F = \int_0^{\pi} \frac{x^{1-\alpha}}{(1+x^2)^{(\alpha+1)/2}}\, dx$ then

$$I(\rho) \le B^{(\alpha+1)/2}\, F\,(1-\rho)^{1-2\alpha} + B^{(\alpha+1)/2}\,(1-\rho)^{1-2\alpha}\, E \int_{\pi}^{\pi/1-\rho} x^{-2\alpha}\, dx .$$

This implies that there is a constant G depending only on α such that

$$I(\rho) \le \begin{cases} G, & \text{when } 0 < \alpha < \tfrac{1}{2} \\ G \log \dfrac{2}{1-\rho}, & \text{when } \alpha = \tfrac{1}{2} \\ \dfrac{G}{(1-\rho)^{2\alpha-1}}, & \text{when } \tfrac{1}{2} < \alpha < 1. \end{cases}$$

Since the integrals $\int_0^1 (1-\rho)^{\alpha-1}\, d\rho$, $\int_0^1 (\log \frac{2}{1-\rho})\,(1-\rho)^{-1/2}\, d\rho$ and

$\int_0^1 (1-\rho)^{-\alpha}\, d\rho$ are finite, the estimates on I(ρ) yield a constant D depending only on α with $J \le D$. This proves (6.40).

Theorem 6.16 *Suppose that the function f is analytic in* $\mathbb{D}$ *and* $0 < \alpha < 1$. *If*

$$I_\alpha(f) \equiv \sup_{|\zeta|=1} \int_0^1 |f'(r\zeta)|\,(1-r)^{\alpha-1}\, dr < \infty \qquad (6.41)$$

then f 0 $\mathcal{M}_\alpha$ *and there is a positive constant* B *depending only on* α *such that*

$$\| f \|_{\mathcal{M}_\alpha} \le B\,(I_\alpha(f) + \| f \|_{H^\infty}) \qquad (6.42)$$

for all such functions f.

Proof: Suppose that f is analytic in $\mathbb{D}$, $0 < \alpha < 1$ and $I_\alpha(f) < \infty$. Theorem 6.13 implies that f extends continuously to $\overline{D}$. Let $|\zeta| = 1$. Then

$$f(z)\,\frac{1}{(1-\overline{\zeta}z)^\alpha} = \frac{f(\zeta)}{(1-\overline{\zeta}z)^\alpha} + g(z)$$

where $g(z) = g(z;\zeta) = \dfrac{f(z) - f(\zeta)}{(1-\overline{\zeta}z)^\alpha}$ $(|z| < 1)$. We shall show that g 0 $\mathcal{F}_\alpha$ and there is a positive constant D depending only on α such that

$$\| g \|_{\mathcal{F}_\alpha} \le D \qquad (6.43)$$

for all $|\zeta| = 1$. Since

$$\left\| \frac{f(\zeta)}{(1-\overline{\zeta}z)^\alpha} \right\|_{\mathcal{F}_\alpha} = |f(\zeta)| \le \| f \|_{H^\infty}$$

Theorem 6.5 yields the conclusion f 0 $\mathcal{M}_\alpha$. We have

$$g'(z) = \frac{f'(z)}{(1-\overline{\zeta}z)^\alpha} + \alpha\overline{\zeta}\,\frac{f(z)-f(\zeta)}{(1-\overline{\zeta}z)^{\alpha+1}}. \qquad (6.44)$$

Hence Theorem 2.14 yields (6.43) if we show that

$$J_\alpha(f) \equiv \sup_{|\zeta|=1} \int_0^1 \int_{-\pi}^{\pi} \frac{|f'(re^{i\theta})|}{|1-\bar{\zeta}re^{i\theta}|^\alpha} (1-r)^{\alpha-1}\, d\theta\, dr < \infty \tag{6.45}$$

and

$$K_\alpha(f) \equiv \sup_{|\zeta|=1} \int_0^1 \int_{-\pi}^{\pi} \frac{|f(re^{i\theta})-f(\zeta)|}{|1-\bar{\zeta}re^{i\theta}|^{\alpha+1}} (1-r)^{\alpha-1}\, d\theta\, dr < \infty. \tag{6.46}$$

We shall show that there is a constant E with $J_\alpha(f) \le E\, I_\alpha(f)$ and $K_\alpha(f) \le E\, I_\alpha(f)$. This also implies (6.42).

Let $\zeta = e^{i\eta}$ where $-\pi < \eta \le \pi$, and let $0 \le r < 1$. Lemma 2.17, part (a), gives $|1-re^{i\gamma}| \ge B|\gamma|$ where B is a positive constant. Hence

$$\begin{aligned}
&\int_0^1 \int_{-\pi}^{\pi} \frac{|f'(re^{i\theta})|}{|1-\bar{\zeta}re^{i\theta}|^\alpha} (1-r)^{\alpha-1}\, d\theta\, dr \\
&= \int_0^1 \int_{-\pi}^{\pi} \frac{|f'(re^{i(\gamma+\eta)})|}{|1-re^{i\gamma}|^\alpha} (1-r)^{\alpha-1}\, d\gamma\, dr \\
&\le \int_0^1 \int_{-\pi}^{\pi} \frac{|f'(re^{i(\gamma+\eta)})|}{B^\alpha |\gamma|^\alpha} (1-r)^{\alpha-1}\, d\gamma\, dr \\
&= \frac{1}{B^\alpha} \int_{-\pi}^{\pi} \frac{1}{|\gamma|^\alpha} \int_0^1 |f'(re^{i(\gamma+\eta)})|\, (1-r)^{\alpha-1}\, dr\, d\gamma \\
&\le \frac{1}{B^\alpha} \int_{-\pi}^{\pi} \frac{1}{|\gamma|^\alpha} I_\alpha(f)\, d\gamma = \frac{2\pi^{1-\alpha}}{B^\alpha(1-\alpha)} I_\alpha(f).
\end{aligned}$$

Therefore

$$J_\alpha(f) \le \frac{2\pi^{1-\alpha}}{B^\alpha(1-\alpha)} I_\alpha(f). \tag{6.47}$$

At each point z where a function h is analytic and nonzero, we have $\left|\frac{\partial}{\partial r}\,|h(z)|\right| \le |h'(z)|$ $(r = |z|)$. We may assume that f is not a constant function and hence its zeros are isolated. Thus an integration by parts yields

$$\int_0^1 \frac{|f(re^{i\theta}) - f(\zeta)|}{|1-\bar{\zeta}re^{i\theta}|^{\alpha+1}}(1-r)^{\alpha-1}\,dr$$

$$\le |f(e^{i\theta}) - f(\zeta)| \int_0^1 \frac{(1-\rho)^{\alpha-1}}{|1-\bar{\zeta}\rho e^{i\theta}|^{\alpha+1}}\,d\rho + \int_0^1 |f'(re^{i\theta})| \int_0^r \frac{(1-\rho)^{\alpha-1}}{|1-\bar{\zeta}\rho e^{i\theta}|^{\alpha+1}}\,d\rho\,dr.$$

The inequalities (6.32) and (6.35) give

$$\int_0^1 \frac{|f(re^{i\theta}) - f(\zeta)|}{|1-\bar{\zeta}re^{i\theta}|^{\alpha+1}}(1-r)^{\alpha-1}\,dr$$

$$\le C\, I_\alpha(f)\, |\theta-\eta|^{1-\alpha} \int_0^1 \frac{(1-\rho)^{\alpha-1}}{|1-\bar{\zeta}\rho e^{i\theta}|^{\alpha+1}}\,d\rho$$

$$+ A \int_0^1 |f'(re^{i\theta})|\,(1-r)^{\alpha-1} \frac{1}{|1-\bar{\zeta}re^{i\theta}|^{\alpha}}\,dr.$$

Hence (6.47) yields

$$\int_0^1 \int_{-\pi}^{\pi} \frac{|f(re^{i\theta}) - f(\zeta)|}{|1-\bar{\zeta}re^{i\theta}|^{\alpha+1}}(1-r)^{\alpha-1}\,d\theta\,dr$$

$$\le 2C\, I_\alpha(f) \int_0^1 \int_0^\pi \frac{\varphi^{1-\alpha}}{|1-\rho e^{i\varphi}|^{\alpha+1}}(1-\rho)^{\alpha-1}\,d\varphi\,d\rho + \frac{2A\,\pi^{1-\alpha}}{B^\alpha(1-\alpha)}\, I_\alpha(f).$$

Thus (6.40) yields $K_\alpha(f) \le (2C\,D + \frac{2A\pi^{1-\alpha}}{B^\alpha(1-\alpha)})\; I_\alpha(f)$, and the proof is complete. □

Corollary 6.17 *Suppose that the function f is analytic in* $\mathbb{D}$*. If* $0 < \alpha < 1$ *and*

$$\int_0^1 \left\{ \max_{|z|=r} |f'(z)| \right\} (1-r)^{\alpha-1} \, dr < \infty$$

then $f \in \mathcal{M}_\alpha$.

Corollary 6.17 is an immediate consequence of Theorem 6.16. Since the condition on f in the corollary depends only on the growth of $|f'|$, this gives concrete examples of functions in $\mathcal{M}_\alpha$ when $0 < \alpha < 1$.

Chapter 7 gives further results about multipliers. Theorem 6.16 plays a role in the proof of some of these results.

NOTES

Lemma 1 is in Duren, Romberg and Shields [1969; see p. 57] in the context of a functional Banach space. A more direct proof that if $f \in \mathcal{M}_\alpha$ for some $\alpha > 0$ then $f \in H^\infty$ (and $\| f \|_{H^\infty} \leq \| f \|_{\mathcal{M}_\alpha}$) is in Hibschweiler and MacGregor [1992; see p. 380]. That reference also contains the proofs of Theorems 5, 6, 7, 8 and 10. The result given by Theorem 4 is known but has not appeared in the literature. Theorem 5 is a basic lemma for obtaining facts about $\mathcal{M}_\alpha$ and its proof is the same as that given for $\alpha = 1$ by Vinogradov, Goluzina and Havin [1970; see p. 30]. The case $\alpha = 1$ of Theorem 10 was proved by Vinogradov [1980]. The theorem of Carathéodory used on p. 117 is in Duren [1983; p. 12]. We thank Richard O'Neil for providing the proof of Theorem 13 (see O'Neil [1995]). O'Neil says that the result was known previously and a variant of it is contained in Zygmund [2002 (Vol. 1); see p. 263]. Theorem 16 was proved by Luo [1995] with an argument not depending on Theorem 13. Corollary 17 was proved by Hallenbeck and Samotij [1995] using a different argument.

CHAPTER 7

Multipliers: Further Results

Preamble. We continue the study of multipliers of $\mathcal{F}_\alpha$. The focus is on finding sufficient conditions for membership in $\mathcal{M}_\alpha$.

The first theorem gives a condition which implies that a bounded function f belongs to $\mathcal{M}_\alpha$ in the case $0 < \alpha \leq 1$. The condition is described in terms of the boundary function $F(\theta)$ and the second difference $D(\theta,\varphi) = F(\theta+\varphi) - 2F(\theta) + F(\theta-\varphi)$. A weighted integrability condition on $D(\theta,\varphi)$ implies f 0 $\mathcal{M}_\alpha$. The proof of this theorem in the case $0 < \alpha < 1$ is a consequence of the basic sufficient condition given by Theorem 6.16. When $\alpha = 1$, the argument depends on showing that a certain Toeplitz operator is bounded on H^∞. We first prove a lemma which relates Toeplitz operators and $\mathcal{M}_1$.

We give a number of applications of Theorem 7.3. For example, if $0 < \alpha < 1$ and if the sequence $\{a_n\}$ $(n = 0, 1, ..)$ satisfies $\sum_{n=1}^{\infty} n^{1-\alpha} |a_n| < \infty$, then the function f belongs to $\mathcal{M}_\alpha$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. We derive conditions on the Taylor coefficients of f which imply that f belongs to $\mathcal{M}_\alpha$ for $\alpha = 1$ and for $\alpha > 1$. Namely, if

$$\sum_{n=0}^{\infty} |a_n| \log(n+2) < \infty$$

then f 0 $\mathcal{M}_1$ and if $\sum_{n=0}^{\infty} |a_n| < \infty$ then f 0 $\mathcal{M}_\alpha$ for $\alpha > 1$. When $\alpha = 1$ the argument depends on a known estimate for $\int_{-\pi}^{\pi} |D_n(\theta)| \, d\theta$ where $D_n(\theta) = 1/2 + \sum_{k=1}^{n} \cos(k\theta)$ is the Dirichlet kernel. The result for $\alpha > 1$ is obtained by first showing that the norms $\left\| z^n/(1-\overline{\zeta}z)^{\alpha} \right\|_{F_\alpha}$ are bounded for $|\zeta| = 1$ and $n = 0, 1, \ldots$. Using the result for $\alpha > 1$, we find that the condition f' 0 H^1 implies that f 0 $\mathcal{M}_\alpha$ for all $\alpha > 0$.

We discuss the question of describing the inner functions in $\mathcal{M}_\alpha$. If f 0 $\mathcal{M}_1$ and f is an inner function, then f is a Blaschke product. Since $\mathcal{M}_\alpha$ δ $\mathcal{M}_\beta$ for $\alpha < \beta$, it follows that each inner function in $\mathcal{M}_\alpha$ for some α, $0 < \alpha < 1$, is a Blaschke product. We give a condition on the zeros of an infinite Blaschke product f which characterizes the Blaschke products with f 0 $\mathcal{M}_1$. In the case $0 < \alpha < 1$, we find a related condition on the zeros of an infinite Blaschke product f which implies that f 0 $\mathcal{M}_\alpha$. For the singular inner function

$$S(z) = \exp\left\{-\frac{1+z}{1-z}\right\} \quad (|z| < 1),$$

we find that S0 $\mathcal{M}_\alpha$ if and only if $\alpha > 1$. The proofs of the last two results depend on lengthy technical arguments for estimating certain integrals.

The question of whether $\mathcal{M}_\alpha \neq \mathcal{M}_\beta$ when $\alpha \neq \beta$ is not completely settled. We make a few observations about this problem.

The results in Chapters 6 and 7 generally assume that $\alpha > 0$. At the end of Chapter 7, we quote a number of facts about $\mathcal{M}_0$.

We begin with a brief discussion of Toeplitz operators. This will lead to a lemma which provides a sufficient condition for membership in $\mathcal{M}_1$.

Let G be a complex-valued function defined on $[-\pi, \pi]$ and assume G 0 $L^2([-\pi, \pi])$. Let $\sum_{n=-\infty}^{\infty} c_n e^{in\theta}$ denote the Fourier series for G, that is

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\theta)\, e^{-in\theta}\, d\theta \tag{7.1}$$

for each integer n. Because G 0 L^2 ([–π, π]), Bessel's inequality gives $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$. It follows that the power series $\sum_{n=0}^{\infty} c_n z^n$ converges for

$|z| < 1$. Also, if we let g be the function

$$g(z) = \sum_{n=0}^{\infty} c_n z^n \tag{7.2}$$

then g 0 H^2. The map P defined by $P(G) = g$ for G 0 L^2 ($[-\pi, \pi]$) is called the orthogonal projection of L^2 ($[-\pi, \pi]$) into H^2.

Suppose that g 0 H^1 and let

$$g(z) = \sum_{n=0}^{\infty} b_n z^n \tag{7.3}$$

for $|z| < 1$. Then $G(\theta) \equiv \lim_{r\to 1-} g(re^{i\theta})$ exists for almost all θ in $[-\pi, \pi]$. Also G 0 L^1 ($[-\pi, \pi]$) and

$$\lim_{r\to 1-} \int_{-\pi}^{\pi} |g(re^{i\theta}) - G(\theta)|\, d\theta = 0. \tag{7.4}$$

Let n be a nonnegative integer and let $0 < r < 1$. Then

$$b_n = \frac{1}{2\pi i} \int_{|w|=r} \frac{g(w)}{w^{n+1}}\, dw$$

and hence

$$b_n = \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} g(re^{i\theta})\, e^{-in\theta}\, d\theta. \tag{7.5}$$

Equation (7.4) implies that

$$\lim_{r\to 1-} \int_{-\pi}^{\pi} g(re^{i\theta})\, e^{-in\theta}\, d\theta = \int_{-\pi}^{\pi} G(\theta)\, e^{-in\theta}\, d\theta$$

and hence (7.5) gives

$$b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\theta)\, e^{-in\theta}\, d\theta \text{ for } n = 0, 1, \ldots .$$

Thus the n-th Taylor coefficient of g equals the n-th Fourier coefficient of G for nonnegative integers n. If n is a negative integer and $0 < r < 1$, then Cauchy's theorem gives

$$\int_{|w|=r} \frac{g(w)}{w^{n+1}}\, dw = 0.$$

Hence (7.4) implies that the n-th Fourier coefficient of G is zero for negative integers n. In particular, this discussion shows that P maps $L^2([-\pi, \pi])$ onto H^2.

Suppose that the complex-valued function ϕ belongs to $L^\infty([-\pi, \pi])$. Define T_ϕ by

$$T_\phi(g) = P(\phi G) \tag{7.6}$$

where g 0 H^2 and $G(\theta) \equiv \lim_{r \to 1-} g(re^{i\theta})$ for almost all θ in $[-\pi, \pi]$. Since g 0 H^2, we have G 0 $L^2([-\pi, \pi])$ and thus ϕG 0 $L^2([-\pi, \pi])$. Thus $P(\phi G)$ 0 H^2, that is, T_ϕ maps H^2 into H^2. It is clear that T_ϕ is linear. The operator T_ϕ is called the Toeplitz operator on H^2 with inducing symbol ϕ.

Suppose that f 0 H^∞ and g 0 H^2. Let $F(\theta)$ and $G(\theta)$ denote the boundary functions for f and g, respectively. Then F and G are defined almost everywhere on $[-\pi, \pi]$ and $\overline{F}G$ 0 $L^2([-\pi, \pi])$. Since the n-th Fourier coefficient of $\overline{F}G$ is

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{F(\theta)}\, G(\theta)\, e^{-in\theta}\, d\theta$$

we have

$$\left(T_{\overline{F}}\, g\right)(z) = \sum_{n=0}^{\infty} c_n z^n$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{F(\theta)}\, G(\theta)\, e^{-in\theta}\, d\theta \right\} z^n$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{F(\theta)}\ G(\theta) \sum_{n=0}^{\infty} \left(e^{-i\theta} z\right)^n \ d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\overline{F(\theta)}\ G(\theta)}{1 - e^{-i\theta} z}\ d\theta$$

for $|z| < 1$. Let $\zeta = e^{i\theta}$, $f(\zeta) = F(\theta)$ and $g(\zeta) = G(\theta)$. Thus

$$\left(T_{\overline{F}}\, g\right)(z) = \frac{1}{2\pi i} \int_T \frac{\overline{f(\zeta)}\, g(\zeta)}{\zeta - z} d\zeta \tag{7.7}$$

for $|z| < 1$.

Recall that $\mathcal{A}$ denotes the space of complex-valued functions g which are analytic in $\mathcal{D}$ and continuous in $\overline{D}$, with norm given by

$$\|g\|_A = \sup_{|z| \le 1} |g(z)|.$$

Also $\mathcal{C}$ denotes the space of complex-valued functions g which are defined and continuous on T, with

$$\|g\|_C = \sup_{|z| = 1} |g(z)|.$$

Lemma 7.1 *Suppose that* f 0 H^∞ *and let* $F(\theta) \equiv \lim_{r \to 1-} f(re^{i\theta})$ *for almost all* θ *in* $[-\pi,\pi]$. *If the Toeplitz operator* $T_{\overline{F}}$ *maps* H^∞ *into* H^∞ *and if the restriction of* $T_{\overline{F}}$ *to* H^∞ *is a bounded operator on* H^∞, *then* f 0 $\mathcal{M}_1$.

Proof: Since $\overline{F}$ 0 $L^\infty([-\pi,\pi])$, $T_{\overline{F}}$ is a linear operator on H^2. By assumption, $T_{\overline{F}}$ maps H^∞ into H^∞ and there is a constant B such that

$$\|T_{\overline{F}}(g)\|_{H^\infty} \le B \|g\|_{H^\infty} \tag{7.8}$$

for all g 0 H^∞.

Let $|\zeta| = 1$. For each r with $0 < r < 1$ define J_r: $H^\infty \to$ | by $J_r(h) = \overline{h(r\zeta)}$ where h 0 H^∞. Then J_r is a linear functional on H^∞ and $|J_r(h)| \le \| h \|_{H^\infty}$ for

$h \in H^\infty$. Let L_r denote the composition $J_r \circ T_{\overline{F}}$. Then L_r is a linear functional on H^∞ and (7.8) gives

$$|L_r(g)| \leq \|T_{\overline{F}}(g)\|_{H^\infty} \leq B\|g\|_{H^\infty}$$

for $g \in H^\infty$.

If $g \in \mathcal{A}$ then $\| g \|_{H^\infty} = \| g \|_{A}$ and hence

$$| L_r(g) | \leq B\|g\|_{A} \tag{7.9}$$

for all $g \in \mathcal{A}$. This implies that if $\hat{L}_r$ denotes the restriction of L_r to $\mathcal{A}$, then $\hat{L}_r$ is a continuous linear functional. If $g \in \mathcal{A}$, then the maximum modulus theorem gives

$$\|g\|_{A} = \max_{|z| \leq 1} | g(z) | = \|g\|_{C} .$$

Thus the Hahn-Banach theorem implies that $\hat{L}_r$ can be extended to a continuous linear functional K_r on $\mathcal{C}$ without increasing the norm. From (7.9) this yields

$$\|K_r\| \leq B. \tag{7.10}$$

By the Riesz representation theorem there exists a measure $\mu_r \in \mathcal{M}$ such that

$$K_r(g) = \int_T g(\sigma)\, d\mu_r(\sigma)$$

for $g \in \mathcal{C}$ and $\|\mu_r\| = \|K_r\|$. This is equivalent to the assertion that

$$K_r(g) = \int_T \overline{g(\sigma)}\, d\mu_r(\sigma) \tag{7.11}$$

for $g \in \mathcal{C}$ where $\mu_r \in \mathcal{M}$ and $\|\mu_r\| = \|K_r\|$. By (7.10) we have

$$\|\mu_r\| \leq B. \tag{7.12}$$

Let k be a nonnegative integer and let $g_k(z) = z^k$ for $|z| = 1$. Then (7.11) yields

$$K_r(g_k) = \int_T \overline{\sigma}^k \, d\mu_r(\sigma). \tag{7.13}$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. Since f is bounded, the discussion starting at (7.3) shows that the Fourier series for F is $\sum_{n=0}^{\infty} a_n e^{in\theta}$. Let j be any integer and let d_j denote the j-th Fourier coefficient of $\overline{F}G_k$ where $G_k(\theta) = g_k(e^{i\theta})$. Then

$$d_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{F(\theta)}\, e^{ik\theta}\, e^{-ij\theta}\, d\theta$$

$$= \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta)\, e^{-i(k-j)\theta}\, d\theta}.$$

Thus $d_j = 0$ if $j > k$ and $d_j = \overline{a_{k-j}}$ if $j \le k$, and it follows that

$$P(\overline{F}G_k) = \sum_{j=0}^{k} \overline{a_{k-j}}\, z^j$$

for $|z| < 1$ and for all nonnegative integers k.

Let $\tilde{g}_k(z) = z^k$ for $|z| \le 1$ and $k = 0, 1, \ldots$. Then

$$L_r(\tilde{g}_k) = J_r\,(T_{\overline{F}}G_k) = \sum_{j=0}^{k} a_{k-j}\, r^j \overline{\zeta}^j. \tag{7.14}$$

Since K_r is an extension of L_r from $\mathcal{A}$ to $\mathcal{C}$, (7.13) and (7.14) yield

$$\sum_{j=0}^{k} a_{k-j}\, r^j\, \overline{\zeta}^j = \int_T \overline{\sigma}^k \, d\mu_r(\sigma). \tag{7.15}$$

The inequality (7.12) and the Banach-Alaoglu theorem imply that there is a sequence $\{r_n\}$ (n = 1, 2, …) and a measure $\mu \in \mathcal{M}$ such that $0 < r_n < 1$, $r_n \to 1$ and $\mu_{r_n} \to \mu$ in the weak* topology. Taking such a limit in (7.15) we obtain

$$\sum_{j=0}^{k} a_{k-j}\, \overline{\zeta}^{j} = \int_T \overline{\sigma}^{k}\, d\mu(\sigma) \tag{7.16}$$

for k = 0, 1, 2, … . Also,

$$\|\mu\| \le B. \tag{7.17}$$

For $|z| < 1$,

$$f(z)\,\frac{1}{1-\overline{\zeta}z} = \sum_{k=0}^{\infty} b_k z^k$$

where $b_k = \sum_{j=0}^{k} a_{k-j}\, \overline{\zeta}^{j}$ $(k = 0, 1, \ldots)$. Hence (7.16) gives

$$b_k = \int_T \overline{\sigma}^{k}\, d\mu(\sigma)$$

and thus

$$f(z)\,\frac{1}{1-\overline{\zeta}z} = \sum_{k=0}^{\infty} \left\{ \int_T \overline{\sigma}^{k}\, d\mu(\sigma) \right\} z^k = \int_T \frac{1}{1-\overline{\sigma}z}\, d\mu(\sigma)$$

for $|z| < 1$. This proves that the function $z \mapsto f(z)\,\frac{1}{1-\overline{\zeta}z}$ belongs to $\mathcal{F}_1$. Since (7.17) holds for all ζ ($|\zeta| = 1$), Theorem 6.5 implies that $f \in \mathcal{M}_1$.

In the case $0 < \alpha < 1$, the argument for Theorem 7.3 depends on the following lemma. The lemma can be proved using an argument similar to that given for Lemma 6.14.

Lemma 7.2 *If* $\beta > -1$ *and* $\gamma > \beta + 1$ *there is a positive constant* A *depending only on* β *and* γ *such that*

$$\int_0^1 \frac{(1-r)^\beta}{|1 - re^{i\theta}|^{\gamma+1}}\, dr \le \frac{A}{|\theta|^{\gamma-\beta}} \tag{7.18}$$

for $0 < |\theta| \le \pi$.

Theorem 7.3 *Suppose that* $0 < \alpha \le 1$ *and* f 0 H^∞. *Let* $F(\theta) = \lim_{r\to 1-} f(re^{i\theta})$ *for almost all* θ *and let*

$$D(\theta, \varphi) = F(\theta+\varphi) - 2F(\theta) + F(\theta-\varphi) \tag{7.19}$$

for appropriate values of θ *and* φ. *If*

$$K_\alpha \equiv \sup_\theta \int_0^\pi \frac{|D(\theta, \varphi)|}{\varphi^{2-\alpha}}\, d\varphi < \infty \tag{7.20}$$

then f 0 $\mathcal{M}_\alpha$. *There is a positive constant* B *depending only on* α *such that*

$$\| f \|_{\mathcal{M}_\alpha} \le B\, (K_\alpha + \| f \|_{H^\infty}) \tag{7.21}$$

for all such functions f.

Proof: Suppose that $0 < \alpha < 1$, f 0 H^∞ and (7.20) holds. The Poisson formula (6.23) applies to f. Differentiation with respect to r on both sides of this formula yields

$$e^{i\theta} f'(re^{i\theta}) = \frac{1}{\pi} \int_{-\pi}^{\pi} Q(r, \varphi-\theta)\, F(\varphi)\, d\varphi \tag{7.22}$$

where

$$Q(r, \varphi) = \frac{(1+r^2)\cos\varphi - 2r}{(1-2r\cos\varphi + r^2)^2}. \tag{7.23}$$

The function Q is an even function of φ, has period 2π and $\int_0^\pi Q(r,\varphi)\,d\varphi = 0$.

Thus (7.22) yields

$$e^{i\theta} f'(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Q(r,\varphi)\,\{F(\theta+\varphi) + F(\theta-\varphi)\}\,d\varphi$$

$$= \frac{1}{\pi}\int_0^{\pi} Q(r,\varphi)\,\{F(\theta+\varphi) + F(\theta-\varphi)\}\,d\varphi$$

$$= \frac{1}{\pi}\int_0^{\pi} Q(r,\varphi)\,\{F(\theta+\varphi) - 2F(\theta) + F(\theta-\varphi)\}\,d\varphi.$$

Thus

$$|f'(re^{i\theta})| \le \frac{1}{\pi}\int_0^{\pi} |Q(r,\varphi)|\,|D(\theta,\varphi)|\,d\varphi. \tag{7.24}$$

Since $(1 + r^2)\cos\varphi - 2r = (1-r)^2 - 2(1+r^2)\sin^2\left(\frac{\varphi}{2}\right)$ and $1-2r\cos\varphi + r^2 = |1-re^{i\varphi}|^2$, (7.23) implies that

$$|Q(r,\varphi)| \le \frac{(1-r)^2 + \varphi^2}{|1-re^{i\varphi}|^4}.$$

Hence (7.24) yields

$$\int_0^1 |f'(re^{i\theta})|\,(1-r)^{\alpha-1}\,dr$$
$$\le \frac{1}{\pi}\int_0^{\pi} I(\varphi)\,|D(\theta,\varphi)|\,d\varphi + \frac{1}{\pi}\int_0^{\pi} J(\varphi)\,|D(\theta,\varphi)|\,d\varphi \tag{7.25}$$

where

$$I(\varphi) = \int_0^1 \frac{(1-r)^{\alpha+1}}{|1-re^{i\varphi}|^4}\, dr \tag{7.26}$$

and

$$J(\varphi) = \varphi^2 \int_0^1 \frac{(1-r)^{\alpha-1}}{|1-re^{i\varphi}|^4}\, dr \tag{7.27}$$

for $0 \le \varphi \le \pi$.

Lemma 7.2 implies that $I(\varphi) \le \dfrac{C}{\varphi^{2-\alpha}}$ and $J(\varphi) \le \dfrac{D}{\varphi^{2-\alpha}}$ for $0 < \varphi \le \pi$ where C and D are positive constants depending only on α. Hence (7.25) yields

$$\int_0^1 |f'(re^{i\theta})|\,(1-r)^{\alpha-1}\, dr \le E \int_0^\pi \frac{|D(\theta,\ \varphi)|}{\varphi^{2-\alpha}}\, d\varphi$$

where E is a positive constant depending only on α. Hence the assumption (7.20) gives

$$\sup_\theta \int_0^1 |f'(re^{i\theta})|\,(1-r)^{\alpha-1}\, dr < \infty.$$

Theorem 6.16 implies that f 0 $\mathcal{M}_\alpha$. This argument also yields (7.21). This proves the theorem when $0 < \alpha < 1$.

Now suppose that f 0 H^∞ and

$$K_1 = \sup_\theta \int_0^\pi \frac{|D(\theta,\varphi|}{\varphi}\, d\varphi < \infty. \tag{7.28}$$

Suppose that g 0 H^∞ and let $t = T_{\overline{F}}(g)$. Then (7.7) gives

$$\sup_{|z|<1} |t(z)| = \sup\left\{ \frac{1}{2\pi}\left| \int_T \frac{\overline{f(\zeta)}\, g(\zeta)}{\zeta - z}\, d\zeta \right| : |z| < 1 \right\}$$

where $f(\zeta) = F(\theta)$ and $g(\zeta) = G(\theta)$. Hence

$$\sup_{|z|<1} |t(z)| = \sup \left\{ \frac{1}{2\pi} \left| \int_T \frac{\overline{f(\zeta)}\; g(\zeta)}{\zeta - r\sigma}\, d\zeta \right| : 0 \le r < 1, |\sigma| = 1 \right\}$$

$$= \sup \left\{ \frac{1}{2\pi} \left| \int_T \frac{\overline{f(\zeta)}\; g(\zeta)}{(1 - r\sigma\bar{\zeta})\zeta}\, d\zeta \right| : 0 \le r < 1, |\sigma| = 1 \right\}$$

$$= \sup \left\{ \frac{1}{2\pi} \left| \int_T \frac{\overline{f(\sigma\zeta)}\; g(\sigma\zeta)}{(1 - r\bar{\zeta})\zeta}\, d\zeta \right| : 0 \le r < 1, |\sigma| = 1 \right\}.$$

Thus

$$\sup_{|z|<1} |t(z)| \le L + M + N \tag{7.29}$$

where

$$L = \sup \left\{ \frac{1}{2\pi} \left| \int_T \frac{\overline{f(\sigma\zeta)} - 2\overline{f(\sigma)} + \overline{f(\sigma\bar{\zeta})}}{(1 - r\bar{\zeta})\zeta}\, g(\sigma\zeta)\, d\zeta \right| \right\}$$

$$M = \sup \left\{ \frac{1}{2\pi} \left| \int_T \frac{2\overline{f(\sigma)}}{(1 - r\bar{\zeta})\zeta}\, g(\sigma\zeta)\, d\zeta \right| \right\}$$

and

$$N = \sup \left\{ \frac{1}{2\pi} \left| \int_T \frac{\overline{f(\sigma\bar{\zeta})}}{(1 - r\bar{\zeta})\zeta}\, g(\sigma\zeta)\, d\zeta \right| \right\}$$

and r and σ vary as before.

Let $\zeta = e^{i\varphi}$ and $\sigma = e^{i\theta}$, where $-\pi < \varphi \le \pi$ and $-\pi < \theta \le \pi$. Let $w = re^{i\psi}$ where $0 \le r < 1$ and $|\psi| \le \pi$. Lemma 2.17, part (a), gives $|1-w| \ge A|\psi|$ for a positive constant A. Hence

$$L \le \|g\|_{H^\infty} \sup\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|F(\theta+\varphi) - 2F(\theta) + F(\theta-\varphi)|}{|1 - re^{-i\varphi}|}\, d\varphi : 0 \le r < 1, |\theta| \le \pi\right\}$$

$$\le \|g\|_{H^\infty} \sup\left\{\frac{1}{\pi}\int_{0}^{\pi} \frac{|D(\theta,\varphi)|}{|1 - re^{-i\varphi}|}\, d\varphi : 0 \le r < 1, |\theta| \le \pi\right\}$$

$$\le \|g\|_{H^\infty} \frac{1}{\pi A} \sup\left\{\int_{0}^{\pi} \frac{|D(\theta,\varphi)|}{\varphi}\, d\varphi : |\theta| \le \pi\right\} = \frac{K_1}{\pi A} \|g\|_{H^\infty} .$$

Also

$$M \le 2 \| f \|_{H^\infty} \sup\left\{\frac{1}{2\pi}\left|\int_T \frac{g(\sigma\zeta)}{(1-r\overline{\zeta})\zeta}\, d\zeta\right| : 0 \le r < 1, |\sigma| = 1\right\}$$

$$= 2 \| f \|_{H^\infty} \sup\left\{\left|\frac{1}{2\pi i}\int_T \frac{g(w)}{w - r\sigma}\, dw\right| : 0 \le r < 1, |\sigma| = 1\right\}.$$

Hence Cauchy's formula yields

$$M \le 2 \| f \|_{H^\infty} \sup\left\{| g(r\sigma)| : 0 \le r < 1, |\sigma| = 1\right\} = 2 \| f \|_{H^\infty} \| g \|_{H^\infty} .$$

For $|w| < 1$ let $h(w) = \overline{f(\overline{w})}$. Then h 0 H^∞. The change of variables $w = \overline{\sigma}\zeta$ and Cauchy's formula give

$$\frac{1}{2\pi i}\int_T \frac{\overline{f(\sigma\overline{\zeta})}}{(1 - r\overline{\zeta})\zeta}\, g(\sigma\zeta)\, d\zeta = \frac{1}{2\pi i}\int_T \frac{h(w)\, g(\sigma^2 w)}{w - r\overline{\sigma}}\, dw$$

$$= h(r\overline{\sigma})\, g(r\sigma)$$

$$= \overline{f(r\sigma)}\, g(r\sigma).$$

Therefore $N \le \| f \|_{H^\infty} \| g \|_{H^\infty}$.

The inequalities for L, M and N and (7.29) yield

$$\sup_{|z|<1} \left|t(z)\right| \le \left(\frac{K_1}{\pi A} + 3 \|f\|_{H^\infty}\right) \|g\|_{H^\infty} . \tag{7.30}$$

Therefore $t \in H^\infty$. Thus $T_{\overline{F}}$ maps H^∞ into H^∞ and (7.30) shows that the restriction of $T_{\overline{F}}$ to H^∞ is a bounded operator. Lemma 7.1 implies $f \in \mathcal{M}_1$.

Corollary 7.4 *Suppose that* $f \in H^\infty$ *and* $0 < \alpha \leq 1$. *If*

$$\sup_{\theta} \int_{-\pi}^{\pi} \frac{|f(e^{i(\theta+\varphi)}) - f(e^{i\theta})|}{|\varphi|^{2-\alpha}}\, d\varphi < \infty \tag{7.31}$$

then $f \in \mathcal{M}_\alpha$.

Corollary 7.5 *Suppose that the function f is analytic in* D *and continuous in* $\overline{D}$. *If the function* $\theta \mapsto f(e^{i\theta})$ *satisfies a Lipschitz condition of order* β *for some* β $(0 < \beta < 1)$ *then* $f \in \mathcal{M}_\alpha$ *for* $\alpha > 1-\beta$.

Proof: By hypothesis there is a constant A such that

$$|f(e^{i(\theta+\varphi)}) - f(e^{i\theta})| \leq A\, |\varphi|^\beta$$

for all θ and φ. Hence if $\alpha > 1-\beta$ then

$$\int_{-\pi}^{\pi} \frac{|f(e^{i(\theta+\varphi)}) - f(e^{i\theta})|}{|\varphi|^{2-\alpha}}\, d\varphi \leq A \int_{-\pi}^{\pi} |\varphi|^{\beta+\alpha-2}\, d\varphi < \infty.$$

Corollary 7.4 yields $f \in \mathcal{M}_\alpha$ in the case $1 - \beta < \alpha \leq 1$. Theorem 6.6 implies that $f \in \mathcal{M}_\alpha$ for all $\alpha > 1-\beta$. □

Suppose that F is a complex-valued function defined on $(-\infty, \infty)$ and F is periodic with period 2π. The modulus of continuity of F is the function ω defined by

$$\omega(t) = \sup_{|x-y| \leq t} |F(x) - F(y)| \quad (t > 0).$$

The next corollary is an immediate consequence of Corollary 7.4 and the inequality

$$|f(e^{i(\theta+\varphi)}) - f(e^{i\theta})| \leq \omega(|\varphi|).$$

Corollary 7.6 *Suppose that the function f is analytic in* D *and continuous in* $\overline{D}$ *and let* ω *be the modulus of continuity of the function* F *defined by*

$F(x) = f(e^{ix})$ *for real* x. *If* $0 < \alpha \le 1$ *and* $\displaystyle\int_0^\pi \frac{\omega(t)}{t^{2-\alpha}}\,dt < \infty$, *then* f 0 M_α.

Theorem 7.7 *Suppose that* $0 < \alpha < 1$ *and the sequence* $\{a_n\}$ $(n = 0, 1, \ldots)$ *satisfies* $\displaystyle\sum_{n=1}^{\infty} n^{1-\alpha}\,|a_n| < \infty$. *If* $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ $(|z| < 1)$, *then* f 0 M_α.

Proof: The assumptions imply that f is analytic in D and continuous in $\overline{D}$. Let ω denote the modulus of continuity of F where $F(x) = f(e^{ix})$ for real x. For $n = 1, 2, \ldots$ let ω_n denote the modulus of continuity of the function g_n where $g_n(x) = e^{inx}$ for real x. Then

$$|g_n(x) - g_n(y)| = 2\,\left|\sin \frac{n(x-y)}{2}\right| \le n\,|x-y|$$

and hence $\omega_n(t) \le nt$. Also $\omega_n(t) \le 2$. Therefore

$$\begin{aligned}\int_0^\pi \frac{\omega_n(t)}{t^{2-\alpha}} &= \int_0^{\pi/n} \frac{\omega_n(t)}{t^{2-\alpha}}\,dt + \int_{\pi/n}^{\pi} \frac{\omega_n(t)}{t^{2-\alpha}}\,dt \\ &\le n \int_0^{\pi/n} \frac{1}{t^{1-\alpha}}\,dt + 2\int_{\pi/n}^{\pi} \frac{1}{t^{2-\alpha}}\,dt \\ &= \frac{\pi^\alpha}{\alpha}\,n^{1-\alpha} + \frac{2}{1-\alpha}\,\pi^{\alpha-1}\,(n^{1-\alpha} - 1) \\ &\le A\,n^{1-\alpha}\end{aligned}$$

where $A = \pi^\alpha/\alpha + 2\pi^{\alpha-1}/(1-\alpha)$. Hence

$$\begin{aligned}\int_0^\pi \frac{\omega(t)}{t^{2-\alpha}}\,dt &\le \int_0^\pi \frac{1}{t^{2-\alpha}} \left\{\sum_{n=1}^{\infty} |a_n|\,\omega_n(t)\right\} dt \\ &= \sum_{n=1}^{\infty} |a_n| \int_0^\pi \frac{\omega_n(t)}{t^{2-\alpha}}\,dt \\ &\le A \sum_{n=1}^{\infty} n^{1-\alpha}\,|a_n| < \infty.\end{aligned}$$

Corollary 7.6 implies that $f \in \mathcal{M}_\alpha$.

Theorem 7.7 is sharp in the following sense.

Theorem 7.8 *Suppose that* $\{\varepsilon_n\}$ $(n = 0, 1, \ldots)$ *is a sequence of positive numbers such that* $\inf_n \varepsilon_n = 0$ *and suppose that* $0 < \alpha < 1$. *Then there is a sequence* $\{a_n\}$ $(n = 0, 1, \ldots)$ *such that* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(|z| < 1)$ *is analytic in* $\mathbb{D}$, $\sum_{n=1}^{\infty} \varepsilon_n n^{1-\alpha} |a_n| < \infty$ *and* $f \notin \mathcal{M}_\alpha$.

Proof: There is a subsequence of $\{\varepsilon_n\}$, say $\{\varepsilon_{n_k}\}$ $(k = 1, 2, \ldots)$ such that $\varepsilon_{n_k} \le \frac{1}{k^3}$ for $k = 1, 2, \ldots$. Let the sequence $\{b_n\}$ $(n = 1, 2, \ldots)$ be defined by $b_{n_k} = k$ for $k = 1, 2, \ldots$ and $b_n = 0$ for all other values of n. Let $a_0 = 0$ and $a_n = n^{\alpha-1} b_n$ for $n = 1, 2, \ldots$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. The power series for f converges for $|z| < 1$ since

$$\begin{aligned} \overline{\lim_{k\to\infty}} \sqrt[n_k]{|a_{n_k}|} &= \overline{\lim_{k\to\infty}} \left(\sqrt[n_k]{n_k}\right)^{\alpha-1} \sqrt[n_k]{k} \\ &= \overline{\lim_{k\to\infty}} \sqrt[n_k]{k} \\ &\le \overline{\lim_{k\to\infty}} \sqrt[n_k]{n_k} = 1. \end{aligned}$$

We have $\sum_{n=1}^{\infty} \varepsilon_n n^{1-\alpha} |a_n| = \sum_{k=1}^{\infty} \varepsilon_{n_k} k \le \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$. For $k = 1, 2, \ldots$, $a_{n_k} = n_k^{\alpha-1} k$ and hence

$$\overline{\lim_{n\to\infty}} \frac{a_n}{n^{\alpha-1}} = \infty.$$

Therefore $a_n \ne O(n^{\alpha-1})$ and hence $f \notin \mathcal{F}_\alpha$. Theorem 6.3 implies that $f \notin \mathcal{M}_\alpha$.

Lemma 7.9 *Suppose that* $\alpha > 0$, $f_n \in \mathcal{F}_\alpha$ for $n = 1, 2, \ldots$ and $f_n(z) \to f(z)$ as

$n \to \infty$ for $|z| < 1$. *If there is a constant* A *with* $\| f_n \|_{\mathcal{F}_\alpha} \le A$ *for* $n = 1, 2, \ldots$, *then* $f \in \mathcal{F}_\alpha$ and $\| f \|_{\mathcal{F}_\alpha} \le A$.

Proof: Let B > A. For each positive integer n there is a measure $\mu_n \in \mathcal{M}$ which represents f_n in $\mathcal{F}_\alpha$ and with $\|\mu_n\| \le B$. By the Banach-Alaoglu theorem there is a subsequence of $\{\mu_n\}$ which we continue to call $\{\mu_n\}$, and $\mu \in \mathcal{M}$ such that $\mu_n \to \mu$ weak* and $\|\mu\| \le B$. We have

$$f_n(z) = \int_T \frac{1}{(1-\overline{\zeta}z)^\alpha}\, d\mu_n(\zeta) \qquad (|z| < 1)$$

for k = 1, 2, … . For each $z \in \mathcal{D}$, the function $\zeta \mapsto \dfrac{1}{(1-\overline{\zeta}z)^\alpha}$ is continuous on T. Therefore

$$\int_T \frac{1}{(1-\overline{\zeta}z)^\alpha}\, d\mu_n(\zeta) \to \int_T \frac{1}{(1-\overline{\zeta}z)^\alpha}\, d\mu(\zeta)$$

as $k \to \infty$. Also, $f_n(z) \to f(z)$ as $n \to \infty$ for $|z| < 1$. Therefore

$$f(z) = \int_T \frac{1}{(1-\overline{\zeta}z)^\alpha}\, d\mu(\zeta)$$

for $|z| < 1$. This shows that $f \in \mathcal{F}_\alpha$ and $\|f\|_{\mathcal{F}_\alpha} \le \|\mu\|$. Hence $\|f\|_{\mathcal{F}_\alpha} \le B$. This holds for every B > A and thus $\|f\|_{\mathcal{F}_\alpha} \le A$. □

Theorem 7.10 *If the sequence* $\{a_n\}$ $(n = 0, 1, \ldots)$ *satisfies*

$$\sum_{n=0}^{\infty} |a_n| \log(n+2) < \infty \tag{7.32}$$

and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(|z| < 1)$, *then* $f \in \mathcal{M}_1$.

Proof: For n = 0, 1, … let $D_n(\theta) = \frac{1}{2} + \sum_{k=1}^{n} \cos k\theta$ and let μ_n denote the measure on T that corresponds to $D_n(\theta)\, d\theta$ on $[-\pi, \pi]$. Then

$$\begin{aligned}\int_T \frac{1}{1-\overline{\zeta}z}\, d\mu_n(\zeta) &= \int_{-\pi}^{\pi} \frac{1}{1-e^{-i\theta}z}\, D_n(\theta)\, d\theta \\ &= \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} e^{-ik\theta} z^k\, D_n(\theta)\, d\theta \\ &= \sum_{k=0}^{\infty} \left\{ \int_{-\pi}^{\pi} e^{-ik\theta}\, D_n(\theta)\, d\theta \right\} z^k \\ &= \sum_{k=0}^{n} \pi z^k .\end{aligned}$$

Therefore

$$\Big\| \sum_{k=0}^{n} z^k \Big\|_{\mathcal{F}_1} \le \frac{1}{\pi} \int_{-\pi}^{\pi} |D_n(\theta)|\, d\theta. \tag{7.33}$$

It is known that $\int_{-\pi}^{\pi} |D_n(\theta)|\, d\theta = \frac{4}{\pi} \log n + O(1)$ as $n \to \infty$. (See Zygmund, vol. I [2002], p. 67). Hence there is a positive constant A such that $\int_{-\pi}^{\pi} |D_n(\theta)|\, d\theta \le A \log(n+2)$ for n = 0, 1, … . Thus (7.33) yields

$$\Big\| \sum_{k=0}^{n} z^k \Big\|_{\mathcal{F}_1} \le \frac{A}{\pi} \log(n+2) \tag{7.34}$$

for n = 0, 1, … . Since

$$\frac{z^n}{1-z} = \frac{1}{1-z} - \sum_{k=0}^{n-1} z^k$$

for n = 1, 2, … and $\| \frac{1}{1-z} \|_{\mathcal{F}_1} = 1$, (7.34) implies that

$$\| \frac{z^n}{1-z} \|_{\mathcal{F}_1} \leq 1 + \frac{A}{\pi} \log(n+1) \tag{7.35}$$

for n = 1, 2, … . Hence there is a positive constant B such that

$$\| \frac{z^n}{1-z} \|_{\mathcal{F}_1} \leq B \log(n+2) \tag{7.36}$$

for n = 0, 1, … . The change of variables $z \mapsto \overline{\zeta} z$ in (7.36) yields

$$\| \frac{z^n}{1-\overline{\zeta} z} \|_{\mathcal{F}_1} \leq B \log(n+2) \tag{7.37}$$

for n = 0, 1, … and $|\zeta| = 1$.

For n = 1, 2, … let $f_n(z) = \sum_{k=0}^{n-1} a_k z^k$. Then (7.37) implies

$$\begin{aligned} \| f_n(z) \frac{1}{1-\overline{\zeta} z} \|_{\mathcal{F}_1} &\leq \sum_{k=0}^{n-1} | a_k | \, \| \frac{z^k}{1-\overline{\zeta} z} \|_{\mathcal{F}_1} \\ &\leq B \sum_{k=0}^{n-1} | a_k | \log(k+2) \\ &\leq B \sum_{k=0}^{\infty} | a_k | \log(k+2) \equiv C. \end{aligned}$$

The assumption (7.32) implies that $C < \infty$. Hence Lemma 7.9 shows that $f(z) \frac{1}{1-\overline{\zeta} z}$ belongs to $\mathcal{F}_1$ and $\| f(z) \frac{1}{1-\overline{\zeta} z} \|_{\mathcal{F}_1} \leq C$ for $|\zeta| = 1$. Theorem 6.5 yields $f \in \mathcal{M}_1$.

We shall show that Theorem 7.10 is precise. The argument uses the following lemma.

Lemma 7.11 *Let* A' *denote the set of functions that are analytic in* $\overline{D}$, *and for*

$g(z) = \sum_{n=0}^{\infty} b_n z^n$ ($|z| < 1$) *let* $c_n = c_n(g) = \sum_{k=0}^{n} b_k$. *Suppose* $f \in H^\infty$ *and let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$. *Then* $f \in \mathcal{M}_1$ *if and only if*

$$\sup \left\{ \left\| \sum_{n=0}^{\infty} a_n c_n(g) z^n \right\|_{H^\infty} : g \in \mathcal{A}', \|g\|_{H^\infty} \leq 1 \right\} < \infty. \tag{7.38}$$

Proof: Suppose that the function f is analytic in $\mathbb{D}$. By Theorem 6.5 and Theorem 2.21, $f \in \mathcal{M}_1$ if and only if there is a positive constant M such that

$$\left\| f(z) \frac{1}{1 - \bar{\zeta} z} * g(z) \right\|_{H^\infty} \leq M \| g \|_{H^\infty}$$

for $g \in H^\infty$ and $|\zeta| = 1$. This is equivalent to the statement that $f \in \mathcal{M}_1$ if and only if

$$\sup \left\{ \left\| f(z) \frac{1}{1 - \bar{\zeta} z} * g(z) \right\|_{H^\infty} : g \in H^\infty, \| g \|_{H^\infty} \leq 1, |\zeta| = 1 \right\} < \infty. \tag{7.39}$$

For $0 < r < 1$ let $g_r(z) = g(rz)$ ($|z| < 1$). If we apply (7.39) to g_r and then let $r \to 1-$, we conclude that $f \in \mathcal{M}_1$ if and only if

$$\sup \left\{ \left\| f(z) \frac{1}{1 - \bar{\zeta} z} * g(z) \right\|_{H^\infty} : g \in \mathcal{A}', \| g \|_{H^\infty} \leq 1, |\zeta| = 1 \right\} < \infty. \tag{7.40}$$

Now suppose that $g \in \mathcal{A}'$ and $\| g \|_{H^\infty} \leq 1$. For $|z| < 1$, let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Let $s_n(z) = \sum_{k=0}^{n} b_k z^k$. Then

$$\frac{1}{1-\bar{\zeta}z}\, f(z) = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} a_k \, \bar{\zeta}^{n-k} \right\} z^n \qquad (|z| < 1).$$

First assume that $f \in \mathcal{A}'$. Then

$$\frac{1}{1-\overline{\zeta}z} f(z) * g(z) = \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} a_k \overline{\zeta}^{n-k}\right\} b_n z^n$$

$$= \sum_{n=0}^{\infty} a_n z^n \left\{\sum_{j=0}^{\infty} b_{n+j}\overline{\zeta}^j z^j\right\}$$

$$= a_0 g(\overline{\zeta}z) + \sum_{n=1}^{\infty} a_n \zeta^n \left[g(\overline{\zeta}z) - s_{n-1}(\overline{\zeta}z)\right]$$

$$= g(\overline{\zeta}z) f(\zeta) - \sum_{n=1}^{\infty} a_n \zeta^n s_{n-1}(\overline{\zeta}z).$$

We have $|g(\overline{\zeta}z) f(\zeta)| \le |f(\zeta)| \le \| f \|_{H^\infty}$. Hence if we first replace f by f_ρ where $f_\rho(z) = f(\rho z)$ $(0 < \rho < 1)$ and let $\rho \to 1-$, it follows that for bounded functions f, $f \in \mathcal{M}_1$ if and only if

$$\sup\left\{\left\|\sum_{n=1}^{\infty} a_n\zeta^n s_{n-1}(\overline{\zeta}z)\right\|_{H^\infty} : g \in \mathcal{A}', \| g \|_{H^\infty} \le 1, |\zeta| = 1\right\} < \infty.$$

This inequality is equivalent to (7.38).

Theorem 7.12 *Let* $\{\varepsilon_n\}$ $(n = 0, 1, \ldots)$ *be a sequence of positive numbers such that* $\inf_n \varepsilon_n = 0$. *There exists a sequence* $\{a_n\}$ $(n = 0, 1, \ldots)$ *such that* $\sum_{n=0}^{\infty} a_n \varepsilon_n \log(n+2) < \infty$ *and the power series* $\sum_{n=0}^{\infty} a_n z^n$ *does not define a member of* $\mathcal{M}_1$.

Proof: There is a subsequence of $\{\varepsilon_n\}$, say $\{\varepsilon_{n_k}\}$ $(k = 0, 1, \ldots)$ such that $n_{k+1} \ge 2n_k$ for all k and $\sum_{k=0}^{\infty} \sqrt{\varepsilon_{n_k}} \le 1$. Let

$$a_n = \frac{1}{\sqrt{\varepsilon_{n_k}}\,\log(n_k + 2)}$$

for $n = n_k$ ($k = 0, 1, \ldots$) and let $a_n = 0$ for all other n. Then

$$\sum_{n=0}^{\infty} |a_n| \, \varepsilon_n \log(n+2) = \sum_{k=0}^{\infty} \sqrt{\varepsilon_{n_k}} < \infty.$$

We may assume that the power series $\sum_{n=0}^{\infty} a_n z^n$ converges for all z with $|z| < 1$. Otherwise, the function defined by the power series has radius of convergence less than 1, and therefore does not give a function in $\mathcal{M}_1$.

For $n = 1, 2, \ldots$ let $P_n(z) = \sum_{j=1}^{n} \frac{1}{j}(z^{n-j} - z^{n+j})$. Then

$$P_n(e^{i\theta}) = \sum_{j=1}^{n} \frac{1}{j}\left[e^{i(n-j)\theta} - e^{i(n+j)\theta}\right] = -2ie^{in\theta} \sum_{j=1}^{n} \frac{\sin(j\theta)}{j}.$$

There is a positive constant A such that $\left|\sum_{j=1}^{n} \frac{\sin(j\theta)}{j}\right| \le A$ for $-\pi \le \theta \le \pi$ and $n = 1, 2, \ldots$. Hence $|P_n(e^{i\theta})| \le 2A$ and therefore

$$|P_n(z)| \le 2A \tag{7.41}$$

for $|z| < 1$ and $n = 1, 2, \ldots$.

Let $g(z) = \sum_{k=0}^{\infty} \sqrt{\varepsilon_{n_k}} \, P_{n_k}(z)$ for $|z| < 1$. If N is any positive integer, then

$$\sum_{n_k \le N} \left| \sqrt{\varepsilon_{n_k}} \, P_{n_k}(z) \right| \le \sum_{n_k \le N} \sqrt{\varepsilon_{n_k}} \, 2A \le 2A \sum_{k=0}^{\infty} \sqrt{\varepsilon_{n_k}} \le 2A.$$

This implies that g is well-defined for $|z| < 1$ and

$$|g(z)| \le 2A \ \text{ for } \ |z| < 1\ . \tag{7.42}$$

Recall from Lemma 7.11 that if $h(z) = \sum_{k=0}^{\infty} b_k z^k$ ($|z| < 1$) then $c_m(h) = \sum_{k=0}^{m} b_k$. The definition of the polynomial P_n implies that $c_m(P_n) \ge 0$ for all m. Since $c_m(g) = \sum_{k=0}^{\infty} \sqrt{\varepsilon_{n_k}}\ c_m(P_{n_k})$ it follows that

$$c_m(g) \ge \sqrt{\varepsilon_{n_k}}\ c_m(P_{n_k})$$

for every k. In particular, if $m = n_k$ then

$$c_{n_k}(g) \ge \sqrt{\varepsilon_{n_k}} \sum_{j=1}^{n_k} \frac{1}{j}.$$

Hence

$$\sum_{n\ge 0} |a_n||c_n(g)| = \sum_{k=0}^{\infty} \frac{1}{\sqrt{\varepsilon_{n_k}}\ \log(n_k + 2)}\, c_{n_k}(g)$$

$$\ge \sum_{k=0}^{\infty} \frac{1}{\log(n_k + 2)} \left\{ \sum_{j=1}^{n_k} \frac{1}{j} \right\}.$$

There is a positive constant B such that $\sum_{j=1}^{N} \frac{1}{j} \ge B \log(N + 2)$ for N = 1, 2, … and thus

$$\sum_{n\ge 0} |a_n||c_n(g)| = \infty. \tag{7.43}$$

We shall show that $f \notin M_1$. To the contrary, suppose that $f \in M_1$. Then f is bounded. For $0 < \rho < 1$ let $g_\rho(z) = g(\rho z)$ for $|z| < 1/\rho$. Lemma 7.11 implies that there is a positive constant C such that $\left\| \sum_{n=0}^{\infty} a_n c_n(g_\rho) z^n \right\|_{H^\infty} \leq C$ for $0 < \rho < 1$. For $|z| < 1$, let

$$F_\rho(z) = \sum_{n=0}^{\infty} a_n c_n(g_\rho) z^n = \sum_{k=0}^{\infty} a_{n_k} c_{n_k}(g_\rho) z^{n_k} .$$

Then F_ρ 0 H^∞ and $\|F_\rho\|_{H^\infty} \leq C$ for $0 < \rho < 1$. A general fact about lacunary series (see Notes) asserts that if $\frac{n_{k+1}}{n_k} > q > 1$ for k = 0, 1, … and if $G(z) = \sum_{k=0}^{\infty} d_k z^{n_k}$ ($|z| < 1$) belongs to H^∞, then there is a positive constant D depending only on q such that $\sum_{k=0}^{\infty} |d_k| \leq D \|G\|_{H^\infty}$. Since $\frac{n_{k+1}}{n_k} \geq 2$, it follows that

$$\sum_{k=0}^{\infty} |a_{n_k} c_{n_k}(g_\rho)| \leq D \| F_\rho \|_{H^\infty}$$

and hence

$$\sum_{k=0}^{\infty} a_{n_k} |c_{n_k}(g_\rho)| \leq D\, C$$

for $0 < \rho < 1$.

If $g(z) = \sum_{n=0}^{\infty} b_n z^n$ ($|z| < 1$) then $c_n(g_\rho) = \sum_{j=0}^{n} b_j \rho^j$. Hence

$$\sum_{k=0}^{\infty} a_{n_k} \left| \sum_{j=0}^{n_k} b_j \rho^j \right| \leq D\, C$$

for $0 < \rho < 1$. Letting $\rho \to 1-$ this yields

$$\sum_{k=0}^{\infty} a_{n_k} \left| \sum_{j=0}^{n_k} b_j \right| \leq D\ C,$$

that is, $\sum_{k=0}^{\infty} | a_{n_k} c_{n_k} (g) | \leq D\ C$. This contradicts (7.43).

Next we show that absolute convergence of the Taylor series for the function f is sufficient to imply that f 0 $\mathcal{M}_\alpha$ for $\alpha > 1$. Two lemmas are used in the proof.

We have noted that if f 0 $\mathcal{F}_\alpha$ ($\alpha > 0$) and if f can be represented by a probability measure in (1.1), then $\|f\|_{\mathcal{F}_\alpha} = 1$. Lemma 7.13 follows from this fact and an application of Lemma 2.5 with $F(z) = \dfrac{1}{(1-z)^\alpha}$ ($|z| < 1$).

Lemma 7.13 *Suppose that* $\alpha > 0$ *and* $|w| \leq 1$, *and let* $f(z) = \dfrac{1}{(1 - wz)^\alpha}$ ($|z| < 1$). *Then* f 0 $\mathcal{F}_\alpha$ *and* $\|f\|_{\mathcal{F}_\alpha} = 1$.

Lemma 7.14 *Suppose that* $\alpha > 1$. *There is a positive constant* B *depending only on* α *such that*

$$\left\| \frac{z^n}{(1-\overline{\zeta}z)^\alpha} \right\|_{\mathcal{F}_\alpha} \leq B \tag{7.44}$$

for $|\zeta| = 1$ *and* $n = 0, 1, \ldots$.

Proof: Let $g_n(z) = \dfrac{z^n}{(1-z)^\alpha}$ for $|z| < 1$ and $n = 0, 1, \ldots$. For $|\zeta| = 1$, $\|g_n(\zeta z)\|_{\mathcal{F}_\alpha} = \|g_n\|_{\mathcal{F}_\alpha}$, and thus (7.44) will follow if we prove that

$$\|g_n\|_{\mathcal{F}_\alpha} \leq B \tag{7.45}$$

for $n = 0, 1, \ldots$. Since $\|g_0\|_{\mathcal{F}_\alpha} = 1$ for $\alpha > 0$, we may assume that $n \geq 1$.

By Theorem 2.10, if f 0 $\mathcal{F}_\alpha$ and $\alpha < \beta$ then f 0 $\mathcal{F}_\beta$ and $\|f\|_{\mathcal{F}_\beta} \leq C\ \|f\|_{\mathcal{F}_\alpha}$ where C depends only on α and β. Thus it suffices to prove (7.45) for $1 < \alpha < 2$ and $n \geq 1$.

For n = 1, 2, … let $r_n = 1 - 1/(2n)$. Then

$$g_n(z) = \frac{1}{(1-z)^\alpha} - \frac{1}{(1-r_n z)^\alpha} + h_n(z) + k_n(z) \tag{7.46}$$

where

$$h_n(z) = (z^n - 1)\left[\frac{1}{(1-z)^\alpha} - \frac{1}{(1-r_n z)^\alpha}\right] \quad (|z| < 1) \tag{7.47}$$

and

$$k_n(z) = \frac{z^n}{(1-r_n z)^\alpha} \qquad (|z| < 1). \tag{7.48}$$

Because of Lemma 7.13 it suffices to prove that

$$\|h_n\|_{F_\alpha} \leq C \tag{7.49}$$

and

$$\|k_n\|_{F_\alpha} \leq D \tag{7.50}$$

for n = 1, 2, … and for suitable constants C and D.

Since $z^n - 1 = (z-1)\sum_{j=0}^{n-1} z^j$, it follows that

$$|z^n - 1| \leq n\,|z - 1| \tag{7.51}$$

for $|z| < 1$ and n = 1, 2, … . Let L denote the closed line segment from $r_n z$ to z. Then

$$\frac{1}{(1-z)^\alpha} - \frac{1}{(1-r_n z)^\alpha} = \int_L \frac{\alpha}{(1-w)^{\alpha+1}}\, dw.$$

If $|z| < 1$, $w = \rho z$ and $0 \leq \rho \leq 1$ then $\dfrac{1}{|1-w|} \leq \dfrac{2}{|1-z|}$. Hence

$$\left|\frac{1}{(1-z)^{\alpha}} - \frac{1}{(1-r_n z)^{\alpha}}\right| \le \frac{\alpha\ 2^{\alpha+1}|z|\,(1-r_n)}{|1-z|^{\alpha+1}}.$$

Therefore

$$\left|\frac{1}{(1-z)^{\alpha}} - \frac{1}{(1-r_n z)^{\alpha}}\right| \le \frac{\alpha\ 2^{\alpha}}{n|1-z|^{\alpha+1}}. \tag{7.52}$$

Since $\dfrac{1}{|1-r_n z|} \le \dfrac{2}{|1-z|}$ we also have

$$\left|\frac{1}{(1-z)^{\alpha}} - \frac{1}{(1-r_n z)^{\alpha}}\right| \le \frac{1+2^{\alpha}}{|1-z|^{\alpha}}. \tag{7.53}$$

We will obtain (7.49) as a consequence of Theorem 2.12. Let

$$I_n = \int_0^1 \int_{-\pi}^{\pi} |h_n(re^{i\theta})|(1-r)^{\alpha-2}\ d\theta\ dr.$$

Then

$$I_n = I_n' + I_n'' + I_n''' \tag{7.54}$$

where

$$I_n' = \int_0^{r_n} \int_{-\pi}^{\pi} |h_n(re^{i\theta})|\ \ (1-r)^{\alpha-2}\ d\theta\ dr \tag{7.55}$$

$$I_n'' = \int_{r_n}^{1} \int_{|\theta|\le 1/(2n)} |h_n(re^{i\theta})|\ \ (1-r)^{\alpha-2}\ d\theta\ dr \tag{7.56}$$

and

$$I_n''' = \int_{r_n}^{1} \int_{1/(2n) \le |\theta| \le \pi} |h_n(re^{i\theta})| \; (1-r)^{\alpha-2} \, d\theta \, dr. \tag{7.57}$$

We estimate I_n' using (7.47), (7.52) and (2.26) as follows.

$$I_n' \le \int_0^{r_n} \left\{ \int_{-\pi}^{\pi} \frac{\alpha \; 2^{\alpha+1}}{n\,|1-re^{i\theta}|^{\alpha+1}} \, d\theta \right\} (1-r)^{\alpha-2} \, dr$$

$$\le \frac{\alpha \; 2^{\alpha+2} \pi}{n} A_{\alpha+1} \int_0^{r_n} (1-r)^{-2} \, dr$$

$$= \frac{\alpha \; 2^{\alpha+2} \, \pi \, A_{\alpha+1} \, r_n}{n(1-r_n)} \le \alpha \, 2^{\alpha+3} \, \pi \, A_{\alpha+1} \, .$$

We estimate I_n'' using (7.51), (7.53) and the inequality $|1-z| \ge A\,|\theta|$ given in Lemma 2.17. This yields

$$I_n'' \le \int_{r_n}^{1} \left\{ \int_{|\theta| \le 1/(2n)} \frac{n\,(1 + 2^{\alpha})}{|1-re^{i\theta}|^{\alpha-1}} \, d\theta \right\} (1-r)^{\alpha-2} \, dr$$

$$\le \frac{2n\,(1+2^{\alpha})}{A^{\alpha-1}} \int_{r_n}^{1} \left\{ \int_0^{1/(2n)} \frac{1}{\theta^{\alpha-1}} \, d\theta \right\} (1-r)^{\alpha-2} \, dr$$

$$= \frac{1 + 2^{\alpha}}{A^{\alpha-1}(2-\alpha)(\alpha-1)}.$$

Next we use (7.52) and the inequality $|1-z| \ge A|\theta|$ to estimate I_n''', as follows.

$$I_n''' \le \int_{r_n}^{1} \left\{ \int_{1/(2n) \le |\theta| \le \pi} \frac{\alpha \, 2^{\alpha+1}}{n\,|1-re^{i\theta}|^{\alpha+1}} \, d\theta \right\} (1-r)^{\alpha-2} \, dr$$

$$\leq \frac{\alpha\, 2^{\alpha+2}}{nA^{\alpha+1}} \int_{r_n}^{1} \left\{ \int_{1/(2n)}^{\pi} \frac{1}{\theta^{\alpha+1}}\, d\theta \right\} (1-r)^{\alpha-2}\, dr$$

$$= \frac{\alpha\, 2^{\alpha+2}}{nA^{\alpha+1}} \int_{r_n}^{1} \frac{1}{\alpha} \left\{ (2n)^{\alpha} - \left(\tfrac{1}{\pi}\right)^{\alpha} \right\} (1-r)^{\alpha-2}\, dr$$

$$\leq \frac{2^{2\alpha+2} n^{\alpha-1}}{A^{\alpha+1}} \int_{r_n}^{1} (1-r)^{\alpha-2}\, dr = \frac{2^{\alpha+3}}{A^{\alpha+1}(\alpha-1)}.$$

The inequalities for I'_n, I''_n and I'''_n and the relation (7.54) yield $I_n \leq E$ where E is a positive constant depending only on α. Theorem 2.12 implies that $h_n \in \mathcal{F}_\alpha$ and (7.49) holds for n = 1, 2, … and $1 < \alpha < 2$.

A second application of Theorem 2.12 will yield (7.50). Let

$$J_n = \int_{0}^{1} \int_{-\pi}^{\pi} |k_n(re^{i\theta})|\, (1-r)^{\alpha-2}\, d\theta\, dr$$

for n = 1, 2, … . Then

$$J_n = J'_n + J''_n \tag{7.58}$$

where

$$J'_n = \int_{0}^{r_n} \int_{-\pi}^{\pi} |k_n(re^{i\theta})|\, (1-r)^{\alpha-2}\, d\theta\, dr \tag{7.59}$$

and

$$J''_n = \int_{r_n}^{1} \int_{-\pi}^{\pi} |k_n(re^{i\theta})|\, (1-r)^{\alpha-2}\, d\theta\, dr. \tag{7.60}$$

From (7.48) and (2.26) we obtain

$$J_n' = \int_0^{r_n} \left\{ \int_{-\pi}^{\pi} \frac{1}{|1-r_n re^{i\theta}|^{\alpha}} \, d\theta \right\} r^n \ (1-r)^{\alpha-2} \ dr$$

$$\leq \int_0^{r_n} \left\{ \int_{-\pi}^{\pi} \frac{1}{|1-re^{i\theta}|^{\alpha}} \, d\theta \right\} r^n \ (1-r)^{\alpha-2} \ dr$$

$$\leq \int_0^{r_n} 2\pi \ A_\alpha \ r^n \ (1-r)^{-1} \ dr$$

$$\leq \frac{2\pi A_\alpha}{1-r_n} \int_0^{r_n} r^n \ dr$$

$$= \frac{2\pi A_\alpha \ r_n^{\,n+1}}{(1-r_n)(n+1)} \leq 4\pi A_\alpha \, .$$

Also, (2.26) yields

$$J_n'' \leq \int_{r_n}^{1} \left\{ \int_{-\pi}^{\pi} \frac{1}{|1-r_n re^{i\theta}|^{\alpha}} \, d\theta \right\} (1-r)^{\alpha-2} \ dr$$

$$\leq \int_{r_n}^{1} \left\{ \int_{-\pi}^{\pi} \frac{1}{|1-r_n e^{i\theta}|^{\alpha}} \, d\theta \right\} (1-r)^{\alpha-2} \ dr$$

$$\leq \frac{2\pi A_\alpha}{(1-r_n)^{\alpha-1}} \int_{r_n}^{1} (1-r)^{\alpha-2} \ dr = \frac{2\pi A_\alpha}{\alpha-1} .$$

The estimates on J_n' and J_r'' and (7.58) yield $J_n \leq F$ for $n = 1, 2, \ldots$ where F is a positive constant depending only on α. Theorem 2.12 implies that k_n 0 $\mathcal{F}_\alpha$ and (7.50) holds for $n = 1, 2, \ldots$ and $1 < \alpha < 2$.

Theorem 7.15 *Suppose that* $\sum_{n=0}^{\infty} |a_n| < \infty$ *and let* $f(z) = \sum_{n=0}^{\infty} a_n z^n \ \ (|z| < 1)$. *Then* f 0 $\mathcal{M}_\alpha$ *for all* $\alpha > 1$.

Proof: For $n = 0, 1, \ldots$ let $f_n(z) = \sum_{k=0}^{n} a_k z^k$. Lemma 7.14 implies that if $\alpha > 1$ and $|\zeta| = 1$ then

$$\left\| f_n(z) \frac{1}{(1-\overline{\zeta}z)^{\alpha}} \right\|_{\mathcal{F}_\alpha} \leq \sum_{k=0}^{n} |a_k| \left\| \frac{z^k}{(1-\overline{\zeta}z)^{\alpha}} \right\|_{\mathcal{F}_\alpha}$$

$$\leq B \sum_{k=0}^{n} |a_k| \leq B \sum_{k=0}^{\infty} |a_k| \equiv C < \infty .$$

Lemma 7.9 implies that $f(z) \dfrac{1}{(1-\overline{\zeta}z)^{\alpha}}$ 0 $\mathcal{F}_\alpha$ and

$$\left\| f(z) \frac{1}{(1-\overline{\zeta}z)^{\alpha}} \right\|_{\mathcal{F}_\alpha} \leq C$$

for $|\zeta| = 1$. Theorem 6.5 now yields f 0 $\mathcal{M}_\alpha$.

Theorem 7.16 *Suppose that* $\sum_{n=0}^{\infty} |a_n| < \infty$ *and* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(|z| < 1)$. *If* f 0 $\mathcal{F}_0$ *then* f 0 $\mathcal{M}_\alpha$ *for all* $\alpha > 0$.

Proof: Let g 0 $\mathcal{F}_\alpha$ where $\alpha > 0$ and let $h = f \cdot g$. Since g 0 $\mathcal{F}_\alpha$ Theorem 2.8 implies that $g' \in \mathcal{F}_{\alpha+1}$. By Theorem 7.15, f 0 $\mathcal{M}_{\alpha+1}$ and thus $f \cdot g'$ 0 $\mathcal{F}_{\alpha+1}$. Since f 0 $\mathcal{F}_0$ Theorem 2.8 implies that f' 0 $\mathcal{F}_1$. Hence Theorem 2.7 yields $f' \cdot g$ 0 $\mathcal{F}_{\alpha+1}$.

Since $f \cdot g'$ 0 $\mathcal{F}_{\alpha+1}$ and $f' \cdot g$ 0 $\mathcal{F}_{\alpha+1}$ it follows that $h' = f \cdot g' + f' \cdot g$ 0 $\mathcal{F}_{\alpha+1}$. Theorem 2.8 yields $h = f \cdot g$ 0 $\mathcal{F}_\alpha$, and thus f 0 $\mathcal{M}_\alpha$.

Theorem 7.17 *If* f' 0 H^1 *then* f 0 $\mathcal{M}_\alpha$ *for all* $\alpha > 0$.

Proof: Suppose that f' 0 H^1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(|z| < 1)$ and let

$$g(z) = f'(z) = \sum_{n=0}^{\infty} b_n z^n \qquad (|z| < 1).$$

Because g 0 H^1 an inequality due to Hardy asserts that $\sum_{n=0}^{\infty} \frac{|b_n|}{n+1} \leq \pi \| g \|_{H^1}$.

Since $b_n = (n+1)\, a_{n+1}$ for $n = 0, 1, \ldots$ this implies that $\sum_{n=0}^{\infty} |a_n| < \infty$. Also $f' \in H^1$ implies that f' 0 $\mathcal{F}_1$ and thus f 0 $\mathcal{F}_0$. Theorem 7.16 yields f 0 $\mathcal{M}_\alpha$ for all $\alpha > 0$.

Next we consider the membership of inner functions in $\mathcal{M}_\alpha$. Every finite Blaschke product is analytic in $\overline{D}$ and hence belongs to $\mathcal{M}_\alpha$ for all $\alpha > 0$. The next theorem shows that the only inner functions in $\mathcal{M}_1$ are Blaschke products. Furthermore, the theorem provides a description of the Blaschke products in $\mathcal{M}_1$.

Theorem 7.18 *If f is an inner function and f 0 $\mathcal{M}_1$, then f is a Blaschke product. An infinite Blaschke product belongs to $\mathcal{M}_1$ if and only if its zeros* $\{z_n\}$ $(n = 0, 1, \ldots)$, *counting multiplicities, satisfy*

$$\sup_{|\zeta|=1} \sum_{n=1}^{\infty} \frac{1-|z_n|}{|1-\overline{\zeta} z_n|} < \infty. \tag{7.61}$$

In a precise way, condition (7.61) asserts that the zeros of the infinite Blaschke product cannot accumulate too frequently toward any particular direction. Theorem 7.18 is a significant result and its proof is long and difficult. We omit the argument. The result is due to Hruščev and Vinogradov [1981].

Suppose that f is an inner function, $0 < \alpha < 1$ and f 0 $\mathcal{M}_\alpha$. Theorem 6.6 implies that f 0 $\mathcal{M}_1$. Hence Theorem 7.18 implies that f is a Blaschke product. The next theorem gives a condition on the zeros of an infinite Blaschke product which implies that the product belongs to $\mathcal{M}_\alpha$.

Theorem 7.19 *Suppose that f is an infinite Blaschke product and the zeros of f are* $\{z_n\}$ $(n = 1, 2, \ldots)$, *counting multiplicities. If* $0 < \alpha < 1$ *and*

$$\sup_{|\zeta|=1} \sum_{n=1}^{\infty} \left[\frac{1-|z_n|}{|1-\overline{\zeta} z_n|} \right]^{\alpha} < \infty \tag{7.62}$$

then f 0 $\mathcal{M}_\alpha$.

It is not known whether the zeros of an infinite Blaschke product which belongs to $\mathcal{M}_\alpha$ for some α, $0 < \alpha < 1$, must satisfy (7.62).

Next we give four technical lemmas needed for the proof of Theorem 7.19.

Lemma 7.20 *Suppose that* $\alpha > 0$. *There is a positive constant* A *depending only on* α *such that*

$$\int_0^1 \frac{1}{(1-xr)^{\alpha+1}(1-r)^{1-\alpha}}\,dr \le \frac{A}{1-x} \tag{7.63}$$

for $0 \le x < 1$.

Proof: Suppose that $\alpha > 0$ and $0 \le x < 1$. Then

$$\begin{aligned}\int_0^1 \frac{1}{(1-xr)^{\alpha+1}(1-r)^{1-\alpha}}\,dr &\le \int_0^x \frac{1}{(1-r)^2}\,dr + \frac{1}{(1-x)^{\alpha+1}}\int_x^1 \frac{1}{(1-r)^{1-\alpha}}\,dr \\ &= \frac{x}{1-x} + \frac{1}{\alpha(1-x)} \\ &< (1+1/\alpha)\,\frac{1}{1-x}.\end{aligned}$$

Lemma 7.21 *Suppose* $0 < \alpha < 1$. *There is a positive constant* B *such that*

$$\int_{-\pi}^{\pi} \frac{1}{|1-xre^{i\theta}|^2\,|1-\overline{\zeta}re^{i\theta}|^\alpha}\,d\theta \le \frac{B}{(1-xr)^{\alpha+1}} \tag{7.64}$$

for $|\zeta| = 1$, $0 \le x \le 1$ *and* $0 \le r < 1$.

Proof: Suppose that $0 < \alpha < 1$, $|\zeta| = 1$, $0 \le x \le 1$ and $0 \le r < 1$. For $-\pi \le \theta \le \pi$ let

$$F(\theta) = \frac{1}{|1-xre^{i\theta}|^2\,|1-\overline{\zeta}re^{i\theta}|^\alpha}.$$

For the function $g(\theta) = 1/|1-xre^{i\theta}|^2$, the symmetrically decreasing rearrangement of g is g itself. Also, the symmetrically decreasing rearrangement of $1/|1-\overline{\zeta}re^{i\theta}|^\alpha$ is $1/|1-re^{i\theta}|^\alpha$. Hence

$$\int_{-\pi}^{\pi} F(\theta)\,d\theta \le \int_{-\pi}^{\pi} F^*(\theta)\,d\theta \tag{7.65}$$

where

$$F^*(\theta) = \frac{1}{|1-xre^{i\theta}|^2 \; |1-re^{i\theta}|^\alpha}$$

for $-\pi \le \theta \le \pi$ (see Notes for a reference). Using (a) of Lemma 2.17 yields

$$\begin{aligned}\int_{-\pi}^{\pi} F^*(\theta)\,d\theta &= \int_{|\theta|\le 1-xr} F^*(\theta)\;d\theta + \int_{1-xr\le|\theta|\le\pi} F^*(\theta)\,d\theta \\ &\le \int_{|\theta|\le 1-xr} \frac{1}{(1-xr)^2}\,\frac{1}{A^\alpha\;|\theta|^\alpha}\,d\theta + \int_{1-xr\le|\theta|\le\pi} \frac{1}{A^{2+\alpha}\;|\theta|^{2+\alpha}}\,d\theta \\ &= \frac{2}{A^\alpha(1-\alpha)(1-xr)^{\alpha+1}} + \frac{2}{(1+\alpha)A^{2+\alpha}}\left[\frac{1}{(1-xr)^{\alpha+1}} - \pi^{-1-\alpha}\right].\end{aligned}$$

Because of (7.65) this yields (7.64) where B depends only on α.

Lemma 7.22 *Suppose that* $0 < \alpha < 1$. *There is a positive constant* C *such that*

$$\begin{aligned}&\int_{-\pi}^{\pi} \frac{1}{|1-xre^{i\theta}|^2\;|1-\overline{\zeta}re^{i\theta}|^\alpha}\,d\theta \\ &\le C\left\{\frac{|\varphi|^{1-\alpha}}{|1-xre^{i(\varphi/4)}|^2} + \frac{1}{(1-xr)\,|1-re^{i(3\varphi/4)}|^\alpha}\right\}\end{aligned} \tag{7.66}$$

for $\zeta = e^{i\varphi}$, $-\pi \le \varphi \le \pi$, $0 \le x < 1$ *and* $0 \le r < 1$.

Proof: We may assume that $\varphi \ge 0$. Let I denote the integral in the statement of the lemma. Then $I = I_1 + I_2$, where

$$I_1 = \int_{|\theta-\varphi|\le(3\varphi/4)} \frac{1}{|1-xre^{i\theta}|^2\;|1-\overline{\zeta}re^{i\theta}|^\alpha}\,d\theta$$

and

$$I_2 = \int_{(3\varphi/4)\le|\theta-\varphi|\le 2\pi} \frac{1}{|1-xre^{i\theta}|^2\;|1-\overline{\zeta}re^{i\theta}|^\alpha}\,d\theta.$$

We proceed to estimate I_1 and I_2. In the second inequality below, we use part (a) of Lemma 2.17.

$$
\begin{aligned}
I_1 &\le \frac{1}{|1-xre^{i\varphi/4}|^2} \int_{|\theta-\varphi|\le(3\varphi)/4} \frac{1}{(1-\overline{\zeta}re^{i\theta}|^\alpha} \, d\theta \\
&\le \frac{1}{A^\alpha |1-xre^{i\varphi/4}|^2} \int_{|\theta-\varphi|\le(3\varphi)/4} \frac{1}{|\theta-\varphi|^\alpha} \, d\theta \\
&= \frac{1}{A^\alpha |1-xre^{i\varphi/4}|^2} \frac{2(3\varphi/4)^{1-\alpha}}{1-\alpha}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
I_2 &\le \frac{1}{|1-re^{i(3\varphi/4)}|^\alpha} \int_{|\theta-\varphi|\ge(3\varphi/4)} \frac{1}{|1-xre^{i\theta}|^2} \, d\theta \\
&\le \frac{1}{|1-re^{i(3\varphi/4)}|^\alpha} \int_{-\pi}^{\pi} \frac{1}{|1-xre^{i\theta}|^2} \, d\theta \\
&= \frac{1}{|1-re^{i(3\varphi/4)}|^\alpha} \frac{2\pi}{1-x^2r^2}.
\end{aligned}
$$

Since $I = I_1 + I_2$, the estimates on I_1 and I_2 yield (7.66), where C depends only on α.

Lemma 7.23 *Suppose that* $0 < \alpha < 1$. *There is a positive constant* D *such that*

$$
\int_0^1 \int_{-\pi}^{\pi} \frac{(1-r)^{\alpha-1}}{|1-\overline{z}re^{i\theta}|^2 \, |1-\overline{\zeta}re^{i\theta}|^\alpha} \, d\theta \, dr \le \frac{D}{|1-\overline{\zeta}z|^\alpha \, (1-|z|)^{1-\alpha}} \qquad (7.67)
$$

for $|\zeta| = 1$ *and* $|z| < 1$.

Proof: The periodicity in θ implies that we may assume z is real and nonnegative. Let $z = x$ where $0 \le x < 1$. Also let $\zeta = e^{i\varphi}$ where $-\pi < \varphi \le \pi$. Let

$$
I = \int_0^1 \int_{-\pi}^{\pi} \frac{(1-r)^{\alpha-1}}{|1-xre^{i\theta}|^2 \, |1-\overline{\zeta}re^{i\theta}|^\alpha} \, d\theta \, dr.
$$

We first consider the case where $|\varphi| \le 1-x$. Then

$$\begin{aligned} |1-\overline{\zeta}x|^2 &= (1-x)^2 + 4\ x\ \sin^2(\varphi/2) \\ &\le (1-x)^2 + 4\ \sin^2(\varphi/2) \\ &\le (1-x)^2 + \varphi^2 \\ &\le 2\ (1-x)^2. \end{aligned}$$

Hence $1/(1-x) \le \sqrt{2}\,/\,|1-\overline{\zeta}x|$. This implies

$$\frac{1}{1-x} \le \frac{2^{\alpha/2}}{|1-\overline{\zeta}x|^{\alpha}\ (1-x)^{1-\alpha}}\,. \tag{7.68}$$

We use (7.64), (7.63) and (7.68) to estimate I, as follows.

$$\begin{aligned} I &\le \int_0^1 \frac{B\ (1-r)^{\alpha-1}}{(1-xr)^{\alpha+1}}\ dr \\ &\le \frac{BA}{1-x} \\ &\le \frac{BA\ 2^{\alpha/2}}{|1-\overline{\zeta}x|^{\alpha}\ (1-x)^{1-\alpha}}\,. \end{aligned}$$

Hence (7.67) holds when $|\varphi| \le 1-x$.

Next assume that $1-x \le |\varphi| \le \pi$. Let $\eta = |\varphi|$. Then Lemma 7.22 and part (a) of Lemma 2.17 give

$$\begin{aligned} I &\le C\left\{\int_0^1 \frac{\eta^{1-\alpha}(1-r)^{\alpha-1}}{|1-xre^{i\varphi/4}|^2}\ dr + \int_0^1 \frac{(1-r)^{\alpha-1}}{(1-xr)\,|1-re^{i(3\varphi/4)}|^{\alpha}}\ dr\right\} \\ &\le C\left\{\eta^{1-\alpha}\int_0^{1-(\eta/\pi)} (1-r)^{\alpha-3}\ dr + \frac{16}{A^2\eta^{1+\alpha}}\int_{1-(\eta/\pi)}^1 (1-r)^{\alpha-1}\ dr\right. \\ &\quad \left. + \left(\frac{4}{3A\eta}\right)^{\alpha}\int_0^1 \frac{(1-r)^{\alpha-1}}{1-xr}\ dr\right\}. \end{aligned}$$

The sum of the first two expressions is less than $\left[\frac{1}{2-\alpha}\frac{1}{\pi^{\alpha-2}}+\frac{16}{A^2\alpha\pi^\alpha}\right]\frac{1}{\eta}$.
Hence Lemma 2.15 yields

$$I \le E\left\{\frac{1}{\eta}+\frac{1}{\eta^\alpha(1-x)^{1-\alpha}}\right\}$$

where E is a constant depending only on α. Since $1 - x \le \eta$ we have

$$|1-\overline{\zeta}x|^2 = (1-x)^2 + 4x\sin^2(\varphi/2) \le 2\eta^2$$

and hence $\frac{1}{\eta} \le \frac{\sqrt{2}}{|1-\overline{\zeta}x|}$. Thus $\left(\frac{1}{\eta}\right)^\alpha \le \frac{2^{\alpha/2}}{|1-\overline{\zeta}x|^\alpha}$. Also

$$\frac{1}{\eta} = \frac{1-x}{\eta}\,\frac{1}{1-x} \le \left(\frac{1-x}{\eta}\right)^\alpha \frac{1}{1-x} \le \frac{2^{\alpha/2}}{|1-\overline{\zeta}x|^\alpha}\cdot\frac{1}{(1-x)^{1-\alpha}}.$$

Therefore

$$I \le E\,\frac{2^{(\alpha/2)+1}}{|1-\overline{\zeta}x|^\alpha\,(1-x)^{1-\alpha}}.$$

This yields (7.67).

We now present the proof of Theorem 7.19. We assume that $0 < \alpha < 1$. Let $\{z_n\}$ $(n = 1, 2, \ldots)$ be a sequence in $\mathcal{D}$ such that

$$\sup_{|\zeta|=1}\sum_{n=1}^{\infty}\left[\frac{1-|z_n|}{|1-\overline{\zeta}z_n|}\right]^\alpha < \infty. \tag{7.69}$$

Since $1-|z_n| \le (1-|z_n|)^\alpha \le 2^\alpha(1-|z_n|)^\alpha/|1-\overline{\zeta}z_n|^\alpha$ for each ζ with $|\zeta| = 1$, the assumption (7.69) implies the Blaschke condition, that is, $\sum_{n=1}^{\infty}(1-|z_n|) < \infty$.

Let f denote the Blaschke product having the zeros $\{z_n\}$, counting multiplicities. We may assume that $f(0) \ne 0$ since each function $z \mapsto z^m$, where m is a positive integer, belongs to $\mathcal{M}_\alpha$ for all $\alpha > 0$. Hence we have

$$f(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1-\overline{z}_n z} \tag{7.70}$$

for $|z| < 1$. Inequality (2.37) shows that

$$|f'(z)| \leq \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{|1-\overline{z}_n z|^2}.$$

Let $|\zeta| = 1$. Lemma 7.23 yields

$$\begin{aligned}
&\int_0^1 \int_{-\pi}^{\pi} |f'(re^{i\theta})| \frac{(1-r)^{\alpha-1}}{|1-\overline{\zeta}re^{i\theta}|^{\alpha}} \, d\theta \, dr \\
&\leq \sum_{n=1}^{\infty} (1-|z_n|^2) \int_0^1 \int_{-\pi}^{\pi} \frac{(1-r)^{\alpha-1}}{|1-\overline{z}_n re^{i\theta}|^2 \, |1-\overline{\zeta}re^{i\theta}|^{\alpha}} \, d\theta \, dr \\
&\leq \sum_{n=1}^{\infty} (1-|z_n|^2) \frac{D}{|1-\overline{\zeta}z_n|^{\alpha} (1-|z_n|)^{1-\alpha}} \\
&\leq 2D \sum_{n=1}^{\infty} \left[\frac{1-|z_n|}{|1-\overline{\zeta}z_n|} \right]^{\alpha}.
\end{aligned}$$

Hence (7.69) yields

$$\sup_{|\zeta|=1} \int_0^1 \int_{-\pi}^{\pi} |f'(re^{i\theta})| \frac{(1-r)^{\alpha-1}}{|1-\overline{\zeta}re^{i\theta}|^{\alpha}} \, d\theta \, dr < \infty. \tag{7.71}$$

For $|\zeta| = 1$ and $|z| < 1$, let $g(z) = f(z) \cdot \dfrac{1}{(1-\overline{\zeta}z)^{\alpha}}$. By Theorem 6.5 it suffices to show that $g \in \mathcal{F}_{\alpha}$ and there is a constant A such that $\|g\|_{\mathcal{F}_{\alpha}} \leq A$ for $|\zeta| = 1$. Since $g(0) = f(0)$ is independent of ζ, Theorem 2.8 shows that we need only prove $g' \in \mathcal{F}_{\alpha+1}$ and $\|g\|_{\mathcal{F}_{\alpha+1}} \leq B$ for $|\zeta| = 1$, where B is a constant.

Since $0 < \alpha < 1$ and $\dfrac{1-|z_n|}{|1-\overline{\zeta}\, z_n|} \leq 1$ for $|\zeta| = 1$, the assumption (7.69) implies that

$$\sup_{|\zeta|=1} \sum_{n=1}^{\infty} \frac{1-|z_n|}{|1-\overline{\zeta} z_n|} < \infty .$$

Hence Theorem 7.18 yields $f \in \mathcal{M}_1$. By Theorem 6.6, $f \in \mathcal{M}_{\alpha+1}$. Thus there is a positive constant C such that

$$\left\| f(z) \frac{1}{(1-\overline{\zeta} z)^{\alpha+1}} \right\|_{\mathcal{F}_{\alpha+1}} \leq C$$

for $|\zeta| = 1$. The inequality (7.71) and Theorem 2.12 yield $f'(z) \dfrac{1}{(1-\overline{\zeta} z)^{\alpha}} \in \mathcal{F}_{\alpha+1}$ and

$$\left\| f'(z) \frac{1}{(1-\overline{\zeta} z)^{\alpha}} \right\|_{\mathcal{F}_{\alpha+1}} \leq E$$

for a positive constant E independent of ζ.

Since $g(z) = f(z) \dfrac{1}{(1-\overline{\zeta} z)^{\alpha}}$,

$$g'(z) = \alpha \overline{\zeta} \; f(z) \frac{1}{(1-\overline{\zeta} z)^{\alpha+1}} + f'(z) \frac{1}{(1-\overline{\zeta} z)^{\alpha}} .$$

The arguments above show that $g' \in \mathcal{F}_{\alpha+1}$ and

$$\| g' \|_{\mathcal{F}_{\alpha+1}} \leq \alpha \; C + E$$

for $|\zeta| = 1$.

As noted in the proof of Theorem 7.19, we may assume that $f(0) \neq 0$ because each monomial belongs to $\mathcal{M}_\alpha$. The inverse problem of factoring out the zeros of a multiplier is considered in Chapter 8.

Let $S(z) = \exp \left[-\dfrac{1+z}{1-z} \right]$ for $|z| < 1$. Then S is an inner function and Theorem 7.18 yields $S \notin \mathcal{M}_1$. Theorem 6.6 yields $S \notin \mathcal{M}_\alpha$ for $\alpha \leq 1$. We complete the information about membership of S in $\mathcal{M}_\alpha$ in the next theorem, by showing that $S \in \mathcal{M}_\alpha$ for $\alpha > 1$. We first prove a simple lemma.

Lemma 7.24 *If* $0 \leq t \leq 1$, s 0 | *and* $|s| < 1$, *then*

$$\left|\frac{1}{1-ts}\right| < \frac{2}{|1-s|}\ .$$

Proof: We may assume that $0 \leq t < 1$. The function $s \mapsto \zeta$ where $\zeta = \dfrac{1-s}{1-ts}$ maps D onto the open disk centered at $1/(1+t)$ and having radius $1/(1+t)$. It follows that if $|s| < 1$, then $|\zeta| < 2/(1+t) \leq 2$.

Theorem 7.25 *The function* S *defined by*

$$S(z) = \exp\left[-\frac{1+z}{1-z}\right] \qquad (|z| < 1)$$

belongs to M_α *for* $\alpha > 1$.

Proof: Because of Theorem 6.6 we may assume $1 < \alpha < 2$. By Theorem 6.5 it suffices to show that $S(z) \cdot \dfrac{1}{(1-\overline{\zeta}z)^\alpha}$ 0 F_α and there is a constant A such that

$$\left\| S(z) \cdot \frac{1}{(1-\overline{\zeta}z)^\alpha} \right\|_{F_\alpha} \leq A \qquad (7.72)$$

for all $|\zeta| = 1$. Lemma 6.25 shows that it is enough to prove (7.72) for ζ in a dense subset of T. In particular, it suffices to prove the result for $|\zeta| = 1$ and $\zeta \neq 1$.

Suppose that $1 < \alpha < 2$. Let $\zeta = e^{i\varphi}$ where $-\pi < \varphi \leq \pi$ and $\varphi \neq 0$, and let $\rho = 1 - (1/5)\,|1-\zeta|^2$. Then $1/5 \leq \rho < 1$. We have

$$S(z) \cdot \frac{1}{(1-\overline{\zeta}z)^\alpha} = \frac{S(\zeta)}{(1-\overline{\zeta}z)^\alpha} - \frac{S(\zeta)}{(1-\rho\overline{\zeta}z)^\alpha} + f(z) + g(z) \qquad (7.73)$$

where

$$f(z) = \left[S(z) - S(\zeta)\right]\left[\frac{1}{(1-\overline{\zeta}z)^\alpha} - \frac{1}{(1-\rho\overline{\zeta}z)^\alpha}\right] \qquad (7.74)$$

and

$$g(z) = \frac{S(z)}{(1-\rho\overline{\zeta}z)^{\alpha}}. \tag{7.75}$$

Since $|S(\zeta)| = 1$, Lemma 7.13 implies that the first two functions on the right-hand side of (7.73) belong to $\mathcal{F}_\alpha$ and have norms equal to 1. Hence it suffices to show that f 0 $\mathcal{F}_\alpha$, g 0 $\mathcal{F}_\alpha$, and there are positive constants B and C depending only on α such that

$$\|f\|_{\mathcal{F}_\alpha} \le B \tag{7.76}$$

and

$$\|g\|_{\mathcal{F}_\alpha} \le C \tag{7.77}$$

for $|\zeta| = 1$ and $\zeta \neq 1$.

Let K denote the closed line segment from $\rho\overline{\zeta}z$ to $\overline{\zeta}z$. Then

$$\left|\frac{1}{(1-\overline{\zeta}z)^{\alpha}} - \frac{1}{(1-\rho\overline{\zeta}z)^{\alpha}}\right| = \left|\int_K \frac{\alpha}{(1-w)^{\alpha+1}}\, dw\right|$$
$$\le \alpha(1-\rho) \max_{w\in K} \frac{1}{|1-w|^{\alpha+1}} .$$

Thus Lemma 7.24 yields

$$\left|\frac{1}{(1-\overline{\zeta}z)^{\alpha}} - \frac{1}{(1-\rho\overline{\zeta}z)^{\alpha}}\right| \le \alpha(1-\rho) \frac{2^{\alpha+1}}{|1-\overline{\zeta}z|^{\alpha+1}} . \tag{7.78}$$

A second application of Lemma 7.24 gives

$$\left|\frac{1}{(1-\overline{\zeta}z)^{\alpha}} - \frac{1}{(1-\rho\overline{\zeta}z)^{\alpha}}\right| \le \frac{1+2^{\alpha}}{|1-\overline{\zeta}z|^{\alpha}} . \tag{7.79}$$

For $-\pi \le \theta \le \pi$ and $0 \le r < 1$, let $F(r,\theta) = |f(re^{i\theta})|(1-r)^{\alpha-2}$, and let

$$I = \int_0^1 \int_{-\pi}^{\pi} F(r,\theta)\, d\theta\, dr.$$

We will show that there is a positive constant B_0 independent of ζ with $I \le B_0$. Theorem 2.12 will then yield (7.76).

Note that

$$I = I_1 + I_2 + I_3 \tag{7.80}$$

where

$$I_1 = \int_0^{\rho} \int_{-\pi}^{\pi} F(r,\theta)\, d\theta\ dr \tag{7.81}$$

$$I_2 = \int_{\rho}^{1} \int_{|\theta-\varphi|\le 1-\rho} F(r,\theta)\, d\theta\ dr \tag{7.82}$$

and

$$I_3 = \int_{\rho}^{1} \int_{1-\rho\le|\theta-\varphi|\le\pi} F(r,\theta)\, d\theta\ dr. \tag{7.83}$$

In the remainder of this argument, we shall use the notation B_1, B_2 etc. to denote positive constants. These are absolute constants or depend only on α, where $1 < \alpha < 2$. Since $|S(z)| \le 1$ and $|S(\zeta)| \le 1$, (7.81), (7.74) and (7.78) give

$$I_1 \le \int_0^{\rho} \left\{ \int_{-\pi}^{\pi} \frac{2\alpha(1-\rho)2^{\alpha+1}}{|1-\overline{\zeta} re^{i\theta}|^{\alpha+1}}\, d\theta \right\} (1-r)^{\alpha-2}\, d\theta\ dr.$$

Hence (2.26) implies

$$I_1 \le B_1(1-\rho) \int_0^{\rho} (1-r)^{-2}\, dr \le B_1. \tag{7.84}$$

Suppose that $z = re^{i\theta}$, $\rho \le r < 1$ and $|\theta-\varphi| \le 1-\rho$. If $|w| \le 1$ and $w \ne 1$ then

$$S'(w) = \frac{-2S(w)}{(1-w)^2}$$

and thus

$$|S'(w)| \le \frac{2}{|1-w|^2}. \qquad (7.85)$$

Let L denote the closed line segment from ζ to z. Then

$$S(z) - S(\zeta) = \int_L S'(w)\, dw$$

and (7.85) gives

$$|S(z) - S(w)| \le 2|z-\zeta| \max_{w\in L} \frac{1}{|1-w|^2}. \qquad (7.86)$$

We claim that (1–ζ)/(1–z) is bounded for |ζ| = 1, ζ ≠ 1 and z in the sector Ψ, where

$$\Psi = \{re^{i\theta}:\ 0 \le r \le 1, |\theta-\varphi| \le 1-\rho\}.$$

To show this we may assume φ > 0. Then φ – (1–ρ) = φ – 2/5 (1–cos φ) > 0. Also, since φ ≤ π, φ + (1–ρ) < 2π – [φ–(1–ρ)]. Therefore $e^{i(\varphi-(1-\rho))}$ is the point in Ψ nearest to 1. Hence if z 0 Ψ, then

$$\left|\frac{1-\zeta}{1-z}\right|^2 \le \frac{|1-\zeta|^2}{2-2\cos[\varphi-(1-\rho)]} = \frac{1-\cos\varphi}{1-\cos\lambda}$$

where λ = φ – 2/5 (1–cos φ). Since

$$\lim_{\varphi\to 0} \frac{1-\cos\varphi}{1-\cos\lambda}$$

exists and equals 1, the function

$$\varphi \mapsto \frac{1-\cos\varphi}{1-\cos\lambda}$$

is bounded for 0 < φ ≤ π. This verifies our claim, and thus

$$\max_{w \in L} \frac{1}{|1-w|^2} \le \frac{B_2}{|1-\zeta|^2}.$$

Using this in (7.86) we obtain

$$|S(z) - S(\zeta)| \le 2B_2 \frac{|re^{i\theta} - \zeta|}{|1-\zeta|^2} \tag{7.87}$$

for $\rho \le r < 1$ and $|\theta - \varphi| \le \rho$.

The inequalities (7.87) and (7.79) imply

$$I_2 \le \int_\rho^1 \int_{|\theta-\varphi| \le 1-\rho} \frac{2B_2 |re^{i\theta} - \zeta|}{|1-\zeta|^2} \frac{1+2^\alpha}{|1-\overline{\zeta} re^{i\theta}|^\alpha} (1-r)^{\alpha-2}\, d\theta\, dr$$

$$= \frac{4B_2(1+2^\alpha)}{|1-\zeta|^2} \int_\rho^1 \left(\int_0^{1-\rho} \frac{1}{|1-re^{i\theta}|^{\alpha-1}}\, d\theta \right) (1-r)^{\alpha-2}\, dr.$$

Hence Lemma 2.17, part (a), gives

$$I_2 \le \frac{B_3}{|1-\zeta|^2} \int_\rho^1 \left(\int_0^{1-\rho} \frac{1}{\theta^{\alpha-1}}\, d\theta \right) (1-r)^{\alpha-2}\, dr.$$

Calculation of these integrals yields

$$I_2 \le \frac{B_3}{|1-\zeta|^2} \frac{1-\rho}{(2-\alpha)(\alpha-1)} = \frac{B_3}{5(2-\alpha)(\alpha-1)}$$

and thus

$$I_2 \le B_4 \tag{7.88}$$

for $|\zeta| = 1$ and $\zeta \ne 1$.

We estimate I_3 using (7.79) and Lemma 2.17, part (a), as follows.

$$
\begin{aligned}
I_3 &\le \int_{\rho}^{1} \int_{1-\rho \le |\theta-\varphi| \le \pi} 2\, \frac{1+2^{\alpha}}{|1-\bar{\zeta} r e^{i\theta}|^{\alpha}} (1-r)^{\alpha-2}\, d\theta\, dr \\
&= 4(1+2^{\alpha}) \int_{\rho}^{1} \left(\int_{1-\rho}^{\pi} \frac{1}{|1-re^{i\theta}|^{\alpha}}\, d\theta \right) (1-r)^{\alpha-2}\, dr \\
&\le B_5 \int_{\rho}^{1} \left(\int_{1-\rho}^{\pi} \frac{1}{|\theta|^{\alpha}}\, d\theta \right) (1-r)^{\alpha-2}\, dr .
\end{aligned}
$$

By evaluating these integrals, we find that

$$I_3 \le B_6 \tag{7.89}$$

where $B_6 = B_5/(\alpha-1)^2$.

The relation (7.80) implies that $I \le B_0$, where B_0 is the sum of the constants given in (7.84), (7.88) and (7.89). Theorem 2.12 yields f 0 F_α and $\| f \|_{F_\alpha} \le (\alpha-1)B_0/2\pi$. Thus (7.76) holds with $B = (\alpha-1)B_0/2\pi$.

Next we prove (7.77). In this argument we use C_1, C_2 etc. to denote positive constants. These are absolute constants or depend only on α, where $\alpha > 1$. For $0 \le r < 1$ and $-\pi \le \theta \le \pi$, let $G(r,\theta) = |g(re^{i\theta})|(1-r)^{\alpha-2}$ and let

$$J = \int_0^1 \int_{-\pi}^{\pi} G(r,\theta)\, d\theta\, dr. \tag{7.90}$$

Let $\sigma = 1 - \frac{1}{2}|1-\zeta|$. Since $|\zeta| = 1$, it follows that $0 \le \sigma \le \rho < 1$. We have

$$J = \sum_{n=1}^{5} J_n \tag{7.91}$$

where

$$J_1 = \int_0^{\sigma} \int_{|\theta| \le 1-r} G(r,\theta)\, d\theta\, dr, \tag{7.92}$$

$$J_2 = \int_0^{\sigma} \int_{1-r\le|\theta|\le\pi} G(r,\theta)\, d\theta\, dr, \tag{7.93}$$

$$J_3 = \int_{\sigma}^{\rho} \int_{|\theta-\varphi|\le 3|1-\zeta|} G(r,\theta)\, d\theta\, dr, \tag{7.94}$$

$$J_4 = \int_{\rho}^{1} \int_{|\theta-\varphi|\le 3|1-\zeta|} G(r,\theta)\, d\theta\, dr, \tag{7.95}$$

and

$$J_5 = \int_{\sigma}^{1} \int_{3|1-\zeta|\le|\theta-\varphi|\le\pi} G(r,\theta)\, d\theta\, dr. \tag{7.96}$$

We proceed to estimate the integrals J_n. In the first argument, the constant C_1 is the constant B in Lemma 2.17, part (b). Also we use the change of variable $x = 1/(C_1(1-r))$.

$$\begin{aligned}
J_1 &= \int_0^{\sigma} \int_{|\theta|\le 1-r} \frac{|S(re^{i\theta})|}{|1-\rho\overline{\zeta}re^{i\theta}|^{\alpha}} (1-r)^{\alpha-2}\, d\theta\, dr \\
&= \int_0^{\sigma} \int_{|\theta|\le 1-r} \frac{\exp\left[-\dfrac{1-r^2}{|1-re^{i\theta}|^2}\right]}{|1-\rho\overline{\zeta}re^{i\theta}|^{\alpha}} (1-r)^{\alpha-2}\, d\theta\, dr \\
&\le \int_0^{\sigma} \int_{|\theta|\le 1-r} \frac{\exp\left[-\dfrac{1-r^2}{C_1(1-r)^2}\right]}{(1-r)^{\alpha}} (1-r)^{\alpha-2}\, d\theta\, dr \\
&\le 2\int_0^{\sigma} \frac{1}{1-r} \exp\left[\frac{-1}{C_1(1-r)}\right] dr \\
&= 2\int_{1/C_1}^{1/(C_1(1-\sigma))} \frac{e^{-x}}{x}\, dx \le 2\int_{1/C_1}^{\infty} \frac{e^{-x}}{x}\, dx < \infty.
\end{aligned}$$

Thus $J_1 \le C_2$, where C_2 is the second integral in the previous line.

Suppose that $z = re^{i\theta}$, $0 \le r \le \sigma$ and $1-r \le |\theta| \le \pi$. Since $r \le \sigma$, we have

$$\frac{1}{2}|1-\zeta| = 1-\sigma \le 1-r \le |\zeta - z| = |1-\overline{\zeta}z|$$

and hence

$$|z-1| \le |z-\zeta| + |\zeta - 1| \le 3|1-\overline{\zeta}z|.$$

Thus Lemma 7.24 gives

$$\frac{1}{|1-\rho\overline{\zeta}z|^{\alpha}} \le \frac{2^{\alpha}}{|1-\overline{\zeta}z|^{\alpha}} \le \frac{6^{\alpha}}{|z-1|^{\alpha}} .$$

An application of Lemma 2.17, part (a), yields

$$\frac{1}{|1-\rho\overline{\zeta}z|^{\alpha}} \le \frac{C_3}{|\theta|^{\alpha}} . \tag{7.97}$$

Since $|S(z)| = \exp\left[-\frac{1-|z|^2}{|1-z|^2}\right]$, Lemma 2.17, part (c), implies

$$|S(z)| \le \exp\left[-\frac{C_4(1-r)}{\theta^2}\right]. \tag{7.98}$$

Inequalities (7.97) and (7.98) yield

$$J_2 \le C_5 \int_0^{\sigma} J_2(r)\ (1-r)^{\alpha-2}\ dr$$

where

$$J_2(r) = \int_{1-r}^{\pi} \exp\left[-\frac{C_4(1-r)}{\theta^2}\right] \frac{1}{\theta^{\alpha}}\ d\theta .$$

The change of variables $x = (C_4 (1-r))^{1/2}/\theta$ shows that

$$J_2(r) \le \frac{C_6}{(1-r)^{(\alpha-1)/2}} \int_0^\infty e^{-x^2} x^{\alpha-2}\, dx.$$

Since $\alpha > 1$, the integral in the previous expression is finite. Hence

$$J_2 \le C_7 \int_0^\sigma (1-r)^{(\alpha-3)/2}\, dr \le C_7 \int_0^1 (1-r)^{(\alpha-3)/2}\, dr.$$

The last integral is finite since $\alpha > 1$.

Suppose that $z = re^{i\theta}$, $\sigma \le r \le \rho$ and $|\theta-\varphi| \le 3|1-\zeta|$. We have $|1-\zeta|^2 = 4\sin^2(\varphi/2) \ge 4\varphi^2/\pi^2$ and hence $|\theta| \le |\theta-\varphi| + |\varphi| \le (3+\pi/2)\,|1-\zeta|$. Thus

$$|1-z|^2 = (1-r)^2 + 4r\sin^2(\theta/2) \le (1-\sigma)^2 + \theta^2 \le C_8\,|1-\zeta|^2 .$$

This implies that

$$|S(z)| = \exp\left[-\frac{1-|z|^2}{|1-z|^2}\right] \le \exp\left[\frac{-(1-r)}{C_8\,|1-\zeta|^2}\right].$$

Therefore

$$J_3 \le \int_\sigma^\rho \int_{|\theta-\varphi|\le 3|1-\zeta|} \exp\left[-\frac{(1-r)}{C_8\,|1-\zeta|^2}\right] \frac{(1-r)^{\alpha-2}}{|1-\rho\bar{\zeta}re^{i\theta}|^\alpha}\, d\theta\, dr$$
$$\le \int_\sigma^\rho \left\{\int_{-\pi}^{\pi} \frac{1}{|1-\rho re^{i\theta}|^\alpha}\, d\theta\right\} \exp\left[-\frac{(1-r)}{C_8\,|1-\zeta|^2}\right] (1-r)^{\alpha-2}\, dr.$$

The inequality (2.26) yields

$$J_3 \le C_9 \int_\sigma^\rho \frac{1}{1-r} \exp\left[-\frac{(1-r)}{C_8\,|1-\zeta|^2}\right] dr.$$

The change of variables $x = (1-r)/C_8|1-\zeta|^2$ shows that

$$J_3 \le C_9 \int_{1/(5C_8)}^{\infty} \frac{1}{x} e^{-x} \, dx < \infty .$$

Since $|S(re^{i\theta})| \le 1$ we have

$$J_4 = \int_{\rho}^{1} \int_{|\theta-\varphi|\le 3|1-\zeta|} \frac{|S(re^{i\theta})|}{|1-\rho\bar{\zeta}re^{i\theta}|^{\alpha}} (1-r)^{\alpha-2} \, d\theta \, dr$$
$$\le \int_{\rho}^{1} \left(\int_{-\pi}^{\pi} \frac{1}{|1-\rho re^{i\theta}|^{\alpha}} \, d\theta \right) (1-r)^{\alpha-2} \, dr .$$

Hence (2.26) gives

$$J_4 \le C_{10} \int_{\rho}^{1} \frac{(1-r)^{\alpha-2}}{(1-\rho r)^{\alpha-1}} \, dr$$
$$\le \frac{C_{10}}{(1-\rho)^{\alpha-1}} \int_{\rho}^{1} (1-r)^{\alpha-2} \, dr = \frac{C_{10}}{\alpha-1} .$$

Finally suppose that $z = re^{i\theta}$, $\sigma \le r < 1$ and $3|1-\zeta| \le |\theta-\varphi| \le \pi$. Since $|1-\zeta|^2 \ge 4\varphi^2/\pi^2$ we have

$$|\varphi| \le \frac{\pi}{2} |1-\zeta| \le \frac{\pi}{6} |\theta-\varphi| .$$

Hence $|\theta| \le |\theta-\varphi| + |\varphi| \le (1 + \pi/6)\, |\theta-\varphi|$. Also,

$$|\theta| > |\theta-\varphi| - |\varphi| \ge (3 - \pi/2)\, |1-\zeta| > |1-\zeta| = 2(1-\sigma) \ge 1-r.$$

Hence Lemma 2.17, part (c), applies, and this yields

$$|S(z)| \le \exp\left[-\frac{1-r^2}{C_{11} \;\theta^2} \right] \le \exp\left[-\frac{(1-r)}{C_{12} \;(\theta-\varphi)^2} \right] .$$

This inequality and Lemma 2.17, part (a), give

$$J_5 \le \int_{\sigma}^{1} \int_{3|1-\zeta| \le |\theta-\varphi| \le \pi} \exp\left[-\frac{1-r}{C_{12}(\theta-\varphi)^2}\right] \frac{(1-r)^{\alpha-2}}{C_{13}\,|\theta-\varphi|^{\alpha}}\, d\theta\, dr$$

$$= \frac{2}{C_{13}} \int_{\sigma}^{1} J_5(r)\,(1-r)^{\alpha-2}\, dr,$$

where

$$J_5(r) = \int_{3|1-\zeta|}^{\pi} \exp\left[-\frac{1-r}{C_{12}\;\theta^2}\right] \frac{1}{\theta^{\alpha}}\, d\theta\;.$$

The change of variable $x = (1-r)/(C_{12}\,\theta^2)$ shows that

$$J_5(r) \le \frac{C_{14}}{(1-r)^{(\alpha-1)/2}} \int_{0}^{\infty} e^{-x}\; x^{(\alpha-3)/2}\; dx\;.$$

This integral is finite since $\alpha > 1$. Therefore

$$J_5 \le C_{15} \int_{\sigma}^{1} (1-r)^{(\alpha-3)/2}\; dr$$

$$\le C_{15} \int_{0}^{1} (1-r)^{(\alpha-3)/2}\; dr.$$

The last integral is finite, since $\alpha > 1$.

The inequalities obtained above for J_n ($n = 1, 2, 3, 4, 5$) and (7.91) yield $J \le C_0$ for all $|\zeta| = 1$, $\zeta \ne 1$, where C_0 is a constant depending only on α for $\alpha > 1$. Theorem 2.12 implies that $g \in \mathcal{F}_\alpha$ and $\|g\|_{\mathcal{F}_\alpha} \le C$ for $|\zeta| = 1$ and $\zeta \ne 1$, where C is a constant independent of ζ.

Theorems 7.18 and 7.19 can be used to give explicit examples of functions in $\mathcal{M}_\alpha$ which are analytic in $\overline{D}\setminus\{1\}$ but fail to be continuous at 1. This is done by constructing a sequence $\{z_n\}$ in D obeying (7.61) or (7.62) if $0 < \alpha < 1$, which has $z = 1$ as its only accumulation point. The function $S(z) = \exp\left[-\frac{1+z}{1-z}\right]$ belongs to $\mathcal{M}_\alpha$ for $\alpha > 1$ and is not continuous in $\overline{D}$.

Next we consider the question of whether $\mathcal{M}_\alpha \ne \mathcal{M}_\beta$ when $\alpha \ne \beta$.

Proposition 7.26 *If* $0 < \alpha < 1$ and $\alpha < \beta$, *then* $\mathcal{M}_\alpha \neq \mathcal{M}_\beta$.

Proof: First assume that $0 < \alpha < \beta < 1$. For $n = 2^k$ $(k = 1, 2, \ldots)$ let $a_n = n^{\beta-1} / (\log n)^2$ and let $a_n = 0$ for all other n. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(|z| < 1)$. Then $\sum_{n=1}^{\infty} n^{1-\beta} |a_n| < \infty$. Theorem 7.7 implies that $f \in \mathcal{M}_\beta$. Since $a_n \neq O(n^{\alpha-1})$, it follows that $f \notin \mathcal{F}_\alpha$. By Theorem 6.3, $f \notin \mathcal{M}_\alpha$.

Next suppose that $0 < \alpha < 1$ and $\beta \geq 1$. Choose γ with $\alpha < \gamma < 1$. The previous argument yields $g \in \mathcal{M}_\gamma$ with $g \notin \mathcal{M}_\alpha$. Since $\gamma < \beta$, Theorem 6.6 implies that $g \in \mathcal{M}_\beta$.

Proposition 7.27 *If* $\alpha > 1$ *then* $\mathcal{M}_\alpha \neq \mathcal{M}_1$.

Proof: Let $\varepsilon_n = \dfrac{1}{\log(n+2)}$ $(n = 0, 1, \ldots)$. Since $\inf_n \varepsilon_n = 0$, Theorem 7.12 provides a sequence $\{a_n\}$ $(n = 0, 1, \ldots)$ for which the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ does not belong to $\mathcal{M}_1$. Since $\sum_{n=0}^{\infty} |a_n| < \infty$, Theorem 7.15 implies that $f \in \mathcal{M}_\alpha$ for all $\alpha > 1$.

The question of whether $\mathcal{M}_\alpha \neq \mathcal{M}_\beta$ when $\alpha \neq \beta$ has not been answered in the case $1 < \alpha < \beta$.

Most of our discussion about $\mathcal{M}_\alpha$ assumed that $\alpha > 0$. A brief and incomplete survey of results about $\mathcal{M}_0$ is given below. References for these results are given in the Notes.

Theorem 7.28 *For each* $\alpha > 0$, $\mathcal{M}_0 \subset \mathcal{M}_\alpha$ *and* $\mathcal{M}_0 \neq \mathcal{M}_\alpha$.

Theorem 7.29 *If* $f' \in H^p$ *for some* $p > 1$ *then* $f \in \mathcal{M}_0$. *There is a function f such that* $f' \in H^1$ *and* $f \notin \mathcal{M}_0$.

Theorem 7.30 *If f is analytic in* $\mathbb{D}$ *and*

$$\int_0^1 \int_{-\pi}^{\pi} \left[\log \frac{1}{1-r} \right] |f''(re^{i\theta})| \, d\theta \, dr < \infty$$

then f 0 M_0.

Theorem 7.31 *Suppose that* f 0 H^∞ *and for almost all* θ *let* $f(e^{i\theta}) = \lim_{r \to 1-} f(re^{i\theta})$. *Also let*

$$D(\theta, \varphi) = f(e^{i(\theta+\varphi)}) - 2f(e^{i\theta}) + f(e^{i(\theta-\varphi)}).$$

If

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|D(\theta, \varphi)|}{\varphi^2} \left[\log \frac{2\pi}{|\varphi|}\right] d\varphi \, d\theta < \infty$$

then f 0 $\mathcal{M}_0$.

Theorem 7.32 *There is a function* f *in* $\mathcal{M}_0$ *such that* f *is not continuous in* $\overline{\mathbb{D}}$.

NOTES

The connections between Toeplitz operators and $\mathcal{M}_1$, such as that given by Lemma 1, go back to the work of Vinogradov in 1974 (see Vinogradov [1980] and Hruščev and Vinogradov [1981]).For an exposition of Toeplitz operators, see Böttcher and Silbermann [1990], Douglas [1972] and Zhu [1990]. Theorem 3 is due to Luo and MacGregor [1998]. Corollaries 4, 5 and 6 and Theorem 7 were obtained by Hallenbeck, MacGregor and Samotij [1996]. Hallenbeck and Samotij [1995] proved that if f is analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$ and $f(e^{i\theta})$ satisfies a Lipschitz condition of order 1, then f 0 $\mathcal{M}_0$. The case $\alpha=1$ of Corollary 6 is due to Vinogradov, Goluzina and Havin [1972]. Theorem 7 was proved independently by Dansereau [1992]. Vinogradov [1980] proved Theorems 10 and 12. The fact about lacunary series quoted in the proof of Theorem 12 is in Zygmund [1968; see p. 247]. Theorem 15 is due to Hallenbeck, MacGregor and Samotij [1996]. Vinogradov [1980] proved that f 0 $\mathcal{M}_1$ if $f' \in H^1$. Another argument for this result which relates to functions of bounded mean oscillation was given by Hibschweiler and MacGregor [1992]. That reference also has a proof of the result of Theorem 17. Hardy's inequality, which is quoted in the proof of Theorem 17, is in Duren [1970; see p. 48]. Theorem 18 was proved by Hruščev and Vinogradov [1981]. Theorems 19 and 25 are due to Hallenbeck, MacGregor and Samotij [1996]. The integral inequality about symmetrically decreasing rearrangements used in the proof of Lemma 21 is in Hardy, Littlewood and Pólya [1967; see p. 278]. Cases of Propositions 26 and 27 were shown by Hibschweiler and MacGregor [1992] and by Hibschweiler and Nordgren [1996]. The facts about $\mathcal{M}_0$ quoted in Theorems 28 through 32 were proved by Hallenbeck and Samotij [1993, 1995, 1996b], except for Theorem 31, which is in Hallenbeck [1998].

CHAPTER 8

Composition

Preamble. We study compositions $f \circ \varphi$ where $f \in \mathcal{F}_\alpha$ and φ is an analytic self-map of $\mathbb{D}$. The main interest is to determine those functions φ for which $f \circ \varphi$ belongs to $\mathcal{F}_\alpha$ for every f in $\mathcal{F}_\alpha$. For such φ, the closed graph theorem implies that the map C_φ defined by $C_\varphi(f) = f \circ \varphi$ is a continuous linear operator on $\mathcal{F}_\alpha$. In this case, we say that φ induces the operator C_φ on $\mathcal{F}_\alpha$.

We show that if φ is a conformal automorphism of $\mathbb{D}$, then φ induces a composition operator on $\mathcal{F}_\alpha$ for every $\alpha > 0$. This fact forms a basis for obtaining other results about composition.

If $\alpha \geq 1$ then every analytic function $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ induces a composition operator on $\mathcal{F}_\alpha$. The argument for this depends on obtaining the extreme points of the closed convex hull of the set of functions that are subordinate to $F_\alpha(z) = \dfrac{1}{(1-z)^\alpha}$ in $\mathbb{D}$, where $\alpha \geq 1$. An application yields the following geometric criterion for membership in $\mathcal{F}_\alpha$ when $\alpha \geq 1$: if f is analytic in $\mathbb{D}$ and $f(\mathbb{D})$ avoids two rays in $\mathbb{C}$ then $f \in \mathcal{F}_\alpha$, where $\pi\alpha$ is the maximum of the angles between the rays. Also we show that if $f(\mathbb{D})$ avoids a ray, then $f \in \mathcal{F}_2$.

We find that if C_φ maps $\mathcal{F}_\alpha$ into $\mathcal{F}_\alpha$ for some $\alpha > 0$, then φ induces a composition operator on $\mathcal{F}_\beta$ for any $\beta > \alpha$. Also, if $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ is analytic and its Taylor coefficients $\{b_n\}$ obey $\sum_{n=1}^{\infty} n\,|b_n| < \infty$, then φ induces a composition operator on $\mathcal{F}_\alpha$ for every $\alpha > 0$. It follows that if the analytic self-map φ extends to $\overline{\mathbb{D}}$ and is sufficiently smooth, then C_φ maps $\mathcal{F}_\alpha$ into itself for all $\alpha > 0$.

This chapter also contains results about the factorization of a function in $\mathcal{F}_\alpha$ in terms of its zeros. It is shown that if $f \in \mathcal{F}_\alpha$, $\alpha > 0$ and $f(z_k) = 0$ where $|z_k| < 1$ ($k = 1, 2, \ldots, n$), then f is the product of the monomials $(z-z_k)$ ($k = 1, 2, \ldots, n$)

with a function g, where $g \in \mathcal{F}_\alpha$. This factorization is equivalent to $f = B \cdot h$, where B is the finite Blaschke product with zeros z_k ($k = 1, 2, \ldots, n$) and $h \in \mathcal{F}_\alpha$. In the case $\alpha = 1$, such a factorization is obtained when f has infinitely many zeros, where B is the infinite Blaschke product with those zeros and $h \in \mathcal{F}_1$. The results about $\mathcal{F}_\alpha$ yield corollaries about factorization of functions in $\mathcal{M}_\alpha$.

A function $\varphi : \mathcal{D} \to \mathcal{D}$ is called a conformal automorphism of $\mathcal{D}$ provided that φ is analytic in $\mathcal{D}$ and φ maps $\mathcal{D}$ one-to-one onto $\mathcal{D}$. Conformal automorphisms of $\mathcal{D}$ are characterized by the form

$$\varphi(z) = c\,\frac{z-b}{1-\overline{b}z} \qquad (|z|<1) \tag{8.1}$$

where $|c| = 1$ and $|b| < 1$.

Theorem 8.1 *Let* $\alpha > 0$ *and let* φ *be a conformal automorphism of* $\mathcal{D}$. *Then* $f \circ \varphi \in \mathcal{F}_\alpha$ *for every* $f \in \mathcal{F}_\alpha$.

Proof: Let $\alpha > 0$ and let $f \in \mathcal{F}_\alpha$. Then

$$f(z) = \int_T \frac{1}{(1-\overline{\zeta}z)^\alpha}\, d\mu(\zeta) \qquad (|z|<1) \tag{8.2}$$

where $\mu \in \mathcal{M}$. Relation (8.1) implies that

$$f\big(\varphi(z)\big) = (1-\overline{b}z)^\alpha \int_T \frac{1}{[1-(c\overline{\zeta}+\overline{b})z/(1+cb\overline{\zeta})]^\alpha}\, \frac{1}{(1+cb\overline{\zeta})^\alpha}\, d\mu(\zeta). \tag{8.3}$$

The mapping $\zeta \mapsto \dfrac{1}{(1+cb\overline{\zeta})^\alpha}$ is bounded on T and thus $\dfrac{1}{(1+cb\overline{\zeta})^\alpha}\, d\mu(\zeta)$ defines a measure $\nu \in \mathcal{M}$. The function F defined by $F(\zeta) = \dfrac{c\overline{\zeta}+\overline{b}}{1+cb\overline{\zeta}}$ for $|\zeta| = 1$ is a homeomorphism of T onto T. Thus (8.3) implies that

$$f(\varphi(z)) = (1-\overline{b}z)^{\alpha} \int_{T} \frac{1}{(1-\overline{s}z)^{\alpha}}\, d\lambda(s)$$

for a measure $\lambda \in \mathcal{M}$. This shows that $f \circ \varphi = g \cdot h$ where g is analytic in $\overline{D}$ and $h \in \mathcal{F}_{\alpha}$. Since $g \in \mathcal{M}_{\alpha}$, it follows that $f \circ \varphi \in \mathcal{F}_{\alpha}$.

Corollary 8.2 *Let* $\alpha > 0$ *and let* $f \in \mathcal{M}_{\alpha}$. *If* φ *is a conformal automorphism of* $\mathbb{D}$, *then* $f \circ \varphi \in \mathcal{M}_{\alpha}$.

Proof: Let $\alpha > 0$ and let φ be given by (8.1). Then φ^{-1} is a conformal automorphism of $\mathbb{D}$ and Theorem 8.1 implies that $g \circ \varphi^{-1} \in \mathcal{F}_{\alpha}$ for every $g \in \mathcal{F}_{\alpha}$. Since $f \in \mathcal{M}_{\alpha}$, it follows that $f \cdot (g \circ \varphi^{-1}) \in \mathcal{F}_{\alpha}$. By Theorem 8.1, $(f \circ \varphi) \cdot g \in \mathcal{F}_{\alpha}$ for every $g \in \mathcal{F}_{\alpha}$. □

Theorem 8.3 *Let* $\alpha > 0$ *and suppose that* $f \in \mathcal{F}_{\alpha}$. *If* $|z_k| < 1$ *and* $f(z_k) = 0$ *for* $k = 1, 2, \ldots, n$ *then*

$$f(z) = \left(\prod_{k=1}^{n} \frac{z-z_k}{1-\overline{z}_k z}\right) g(z) \qquad (|z|<1) \tag{8.4}$$

where $g \in \mathcal{F}_{\alpha}$. .

Proof: First suppose that $n = 1$. Suppose that $\alpha > 0$, $f \in \mathcal{F}_{\alpha}$ and $f(z_1) = 0$ for some $z_1 \in \mathbb{D}$. For $|z| < 1$ and $z \neq z_1$, let

$$g(z) = f(z) \div \frac{z-z_1}{1-\overline{z}_1 z}.$$

The function g has a removable singularity at z_1 and thus g is analytic in $\mathbb{D}$. Since (8.4) is clear, we need only show that $g \in \mathcal{F}_{\alpha}$.

For $|z| < 1$, let $h(z) = f\left(\dfrac{z+z_1}{1+\overline{z}_1 z}\right)$. By Theorem 8.1, $h \in \mathcal{F}_{\alpha}$. Let $h(z) = \sum_{n=0}^{\infty} b_n z^n \ (|z|<1)$. Then $b_0 = 0$ and there exists $\mu \in \mathcal{M}$ such that

$$b_n = A_n(\alpha) \int_{T} \overline{\zeta}^{n}\, d\mu(\zeta) \tag{8.5}$$

for n = 1,2, … . Let $j(z) = \dfrac{h(z)}{z}$ for $0 < |z| < 1$. Then j extends analytically to 0 and thus j is analytic in $\mathcal{D}$. We claim that $j \; 0 \; \mathcal{F}_\alpha$. Let $j(z) = \sum_{n=0}^{\infty} c_n z^n$ for $|z| < 1$. To prove the claim, we require $\upsilon \; 0 \; \mathcal{M}$ such that

$$c_n = A_n(\alpha) \int_T \overline{\zeta}^n \, d\upsilon(\zeta) \qquad (8.6)$$

for n = 0,1, … . Since $c_n = b_{n+1}$ for n = 0,1, … and since

$$A_n(\alpha) = \alpha(\alpha+1) \cdots (\alpha+n-1) \ni n!$$

the relations (8.5) and (8.6) show that the condition on υ is

$$\int_T \overline{\zeta}^n \, d\upsilon(\zeta) = \left(1 + \frac{\alpha-1}{n+1}\right) \int_T \overline{\zeta}^{n+1} \, d\mu(\zeta) \qquad (8.7)$$

for n = 0,1, … . Let $m(z) = \sum_{n=0}^{\infty} d_n z^n$ ($|z| < 1$) where $d_n = \int_T \overline{\zeta}^{n+1} \, d\mu(\zeta)$. Let $\lambda \; 0 \; \mathcal{M}$ be defined by $d\lambda(\zeta) = \overline{\zeta} \, d\mu(\zeta)$. Then $d_n = \int_T \overline{\zeta}^n \, d\lambda(\zeta)$ for n = 0, 1, … . Let $n(z) = \sum_{n=0}^{\infty} e_n z^n$ ($|z| < 1$) where $e_n = \dfrac{\alpha-1}{n+1} \int_T \overline{\zeta}^{n+1} \, d\mu(\zeta)$. Then $n \; 0 \; H^2$. Since $H^2 \; \delta \; H^1 \; \delta \; \mathcal{F}_1$, there exists $\sigma \; 0 \; \mathcal{M}$ with $e_n = \int_T \overline{\zeta}^n \, d\sigma(\zeta)$ for n = 0, 1, … . Let υ be defined by $d\upsilon(\zeta) = d\lambda(\zeta) + d\sigma(\zeta)$. Then $\upsilon \; 0 \; \mathcal{M}$ and (8.7) holds. Therefore $j \; 0 \; \mathcal{F}_\alpha$.

The argument above shows that

$$f\left(\frac{z+z_1}{1+\overline{z}_1 z}\right) = z \, j(z) \qquad (|z| < 1).$$

Let $w = \dfrac{z+z_1}{1+\overline{z}_1 z}$ for $|z| < 1$. Then

$$f(\mathrm{w}) = \frac{\mathrm{w}-\mathrm{z}_1}{1-\overline{\mathrm{z}}_1\mathrm{w}}\; j\left(\frac{\mathrm{w}-\mathrm{z}_1}{1-\overline{\mathrm{z}}_1\mathrm{w}}\right)$$

for |w| < 1. By Theorem 8.1 the function $\mathrm{w} \mapsto j\left(\frac{\mathrm{w}-\mathrm{z}_1}{1-\overline{\mathrm{z}}_1\mathrm{w}}\right)$ belongs to $\mathcal{F}_\alpha$, that is, g 0 $\mathcal{F}_\alpha$. This completes the proof in the case n = 1. The general result follows by n applications of the case n = 1.

Corollary 8.4 *Let* α > 0 *and suppose that* f 0 $\mathcal{F}_\alpha$. If $|\mathrm{z}_\mathrm{k}| < 1$ *and* $f(\mathrm{z}_\mathrm{k}) = 0$ for k = 1, 2, …, n *then*

$$f(\mathrm{z}) = \left\{\prod_{\mathrm{k}=1}^{\mathrm{n}} (\mathrm{z}-\mathrm{z}_\mathrm{k})\right\} h(\mathrm{z}) \qquad (|\mathrm{z}| < 1) \qquad (8.8)$$

and h 0 $\mathcal{F}_\alpha$.

Proof: The assumptions imply (8.4) where g 0 $\mathcal{F}_\alpha$. Since the function $\mathrm{z} \mapsto \prod_{\mathrm{k}=1}^{\mathrm{n}} \frac{1}{1-\overline{\mathrm{z}}_\mathrm{k}\,\mathrm{z}}$ is analytic in $\overline{D}$, it belongs to $\mathcal{M}_\alpha$ for every α > 0. Thus the function

$$h(\mathrm{z}) = \left\{\prod_{\mathrm{k}=1}^{\mathrm{n}} \frac{1}{1-\overline{\mathrm{z}}_\mathrm{k}\,\mathrm{z}}\right\} g(\mathrm{z}) \qquad (|\mathrm{z}| < 1)$$

belongs to $\mathcal{F}_\alpha$, and (8.8) holds.

Corollary 8.5 *Suppose that* f 0 $\mathcal{M}_\alpha$ *for some* α > 0. *If* $|\mathrm{z}_\mathrm{k}| < 1$ *and* $f(\mathrm{z}_\mathrm{k}) = 0$ *for* k = 1,2, … , n, *then*

$$f(\mathrm{z}) = \left(\prod_{\mathrm{k}=1}^{\mathrm{n}} \frac{\mathrm{z}-\mathrm{z}_\mathrm{k}}{1-\overline{\mathrm{z}}_\mathrm{k}\mathrm{z}}\right) g(\mathrm{z}) \qquad (|\mathrm{z}| < 1)$$

where g 0 $\mathcal{M}_\alpha$.

Proof: For |z| < 1 let $\mathrm{B}(\mathrm{z}) = \prod_{\mathrm{k}=1}^{\mathrm{n}} \frac{\mathrm{z}-\mathrm{z}_\mathrm{k}}{1-\overline{\mathrm{z}}_\mathrm{k}\mathrm{z}}$ and let $g(\mathrm{z}) = f(\mathrm{z})$ ∋ B(z) for $\mathrm{z} \neq \mathrm{z}_\mathrm{k}$ (k = 1,2, …, n). Then g extends to each z_k and g is analytic in D. It remains to show that g 0 $\mathcal{M}_\alpha$.

Let h 0 $\mathcal{F}_\alpha$. Then $g \cdot h = \dfrac{f \cdot h}{B}$. Since f 0 $\mathcal{M}_\alpha$, $f \cdot h$ 0 $\mathcal{F}_\alpha$. Also $(f \cdot h)(z_k) = 0$ for k = 1,2, …, n. By Theorem 8.3, $\dfrac{f \cdot h}{B}$ 0 $\mathcal{F}_\alpha$. Thus $g \cdot h$ 0 $\mathcal{F}_\alpha$ for every h 0 $\mathcal{F}_\alpha$.

Corollary 8.6 *Suppose that* f 0 $\mathcal{M}_\alpha$ *and* $\alpha > 0$. *If* $|z_k| < 1$ and $f(z_k) = 0$ for $k = 1, 2, \ldots, n$ *then*

$$f(z) = \left\{ \prod_{k=1}^{n} (z - z_k) \right\} h(z) \qquad (|z| < 1) \qquad (8.9)$$

and h 0 $\mathcal{M}_\alpha$.

Proof: This corollary is a consequence of Corollary 8.5 because the product

$$z \longmapsto \prod_{k=1}^{n} \frac{1}{1 - \bar{z}_k z}$$

belongs to $\mathcal{M}_\alpha$ for every $\alpha > 0$.

The next theorem gives a factorization result for f 0 $\mathcal{F}_1$ when f has infinitely many zeros. In the case $\alpha \neq 1$, such factorizations are unknown. If $\alpha > 1$, it is not clear what form such a factorization could take. This is due to the fact that for such α, there are functions $f \neq 0$ with f 0 $\mathcal{F}_\alpha$ such that the zeros of f do not satisfy the Blaschke condition (see Theorem 5.6).

Theorem 8.7 *Suppose that* f 0 $\mathcal{F}_1$, f *has a zero of order* m *at* 0, *and the zeros of* f *in* $\{z: 0 < |z| < 1\}$ *are given by* $\{z_k\}$ $(k = 1, 2, \ldots)$, *listed by multiplicity. For* $|z| < 1$ *let*

$$B(z) = z^m \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \bar{z}_k z}. \qquad (8.10)$$

Then $f = B \cdot g$ *where* g 0 $\mathcal{F}_1$.

Proof: Recall that $\mathcal{F}_1$ δ H^p for $0 < p < 1$. It follows that the infinite product in (8.10) converges for $|z| < 1$. Let

$$f(z) = \int_{T} \frac{1}{1-\overline{\zeta}z}\, d\mu(\zeta) \qquad (|z| < 1) \qquad (8.11)$$

where $\mu \in \mathcal{M}$. Suppose that $|b| < 1$. If $|z| < 1$ and $z \neq b$ then (8.11) yields

$$\frac{f(z)}{(z-b)/(1-\bar{b}z)} - \int_T \frac{1}{(1-\bar{\zeta}z)} \frac{1}{(\zeta-b)/(1-\bar{b}\zeta)}\, d\mu(\zeta) = \frac{1-|b|^2}{z-b} \int_T \frac{1}{1-\bar{\zeta}b}\, d\mu(\zeta). \tag{8.12}$$

Assume that $f(b) = 0$ and let

$$g_0(z) = \int_T \frac{1}{1-\bar{\zeta}z} \frac{1}{(\zeta-b)/(1-\bar{b}\zeta)}\, d\mu(\zeta) \qquad (|z| < 1). \tag{8.13}$$

Then g_0 is analytic in $\mathcal{D}$ and (8.12) yields

$$f(z) = \frac{z-b}{1-\bar{b}z}\, g_0(z) \tag{8.14}$$

for all z with $|z| < 1$ and $z \neq b$. The relation (8.14) is clearly valid for $z = b$, and thus it holds for all $z \in \mathcal{D}$. Since $\left|\frac{\zeta-b}{1-\bar{b}\zeta}\right| = 1$ for $|\zeta| = 1$, (8.13) implies that $g_0(z) = \int_T \frac{1}{1-\bar{\zeta}z}\, d\upsilon(\zeta)$ for some $\upsilon \in \mathcal{M}$ with $\|\upsilon\| = \|\mu\|$. Thus $g_0 \in \mathcal{F}_1$ and

$$\|g_0\|_{\mathcal{F}_1} \leq \|f\|_{\mathcal{F}_1}. \tag{8.15}$$

For $n = 1, 2, \ldots$ let

$$B_n(z) = z^m \prod_{k=1}^{n} \frac{|z_k|}{z_k} \frac{z_k - z}{1-\bar{z}_k z} \qquad (|z| < 1).$$

Applying the previous argument (m + n) times we conclude that $f = B_n \cdot g_n$, where $g_n \in \mathcal{F}_1$ and $\|g_n\|_{\mathcal{F}_1} \leq \|f\|_{\mathcal{F}_1}$. Let $g = f/B$. Since $B_n \to B$ as $n \to \infty$ and $\|g_n\|_{\mathcal{F}_1} \leq \|f\|_{\mathcal{F}_1}$ for $n = 1,2, \ldots$, Lemma 7.10 implies that $g = f/B \in \mathcal{F}_1$.

The proof of the next corollary is identical to the proof of Corollary 8.5.

Corollary 8.8 *Suppose that* f 0 $\mathcal{M}_1$, *f has a zero of order* m *at zero, and the zeros of f in* {z: $0 < |z| < 1$} *are given by* $\{z_k\}$ (k = 1,2,…), *listed by multiplicity. Let* B *be defined by* (8.10). *Then* $f = B \cdot g$ *where* g 0 $\mathcal{M}_1$.

In order to show that each analytic function $\varphi : \mathcal{D} \to \mathcal{D}$ induces a composition operator on $\mathcal{F}_\alpha$ when $\alpha \geq 1$, we develop certain results about extreme points. Let X be a linear topological space. If $\mathcal{F}$ δ X and f 0 $\mathcal{F}$, we call f an extreme point of $\mathcal{F}$ provided that f cannot be written as a proper convex combination of two distinct elements in $\mathcal{F}$. Let E$\mathcal{F}$ denote the set of extreme points of $\mathcal{F}$. Also let H$\mathcal{F}$ denote the closed convex hull of $\mathcal{F}$.

Recall that $\mathcal{P}$ denotes the set of functions f that are analytic in $\mathcal{D}$ such that Re $f(z) > 0$ for $|z| < 1$ and $f(0) = 1$. Clearly $\mathcal{P}$ is convex.

Lemma 8.9 *The set of extreme points of* $\mathcal{P}$ *is*

$$\left\{ f : f(z) = \frac{1+\overline{\zeta}z}{1-\overline{\zeta}z}, \; |\zeta| = 1 \right\}. \tag{8.16}$$

Proof: By Theorem 1.1, functions f 0 $\mathcal{P}$ are characterized by the formula

$$f(z) = \int_T \frac{1+\overline{\zeta}z}{1-\overline{\zeta}z}\, d\mu(\zeta) \qquad (|z| < 1) \tag{8.17}$$

where μ 0 $\mathcal{M}^*$. Corollary 1.6 asserts that the map $\mu \longmapsto f$ from $\mathcal{M}^*$ to $\mathcal{P}$ given by (8.17) is one-to-one. Thus f 0 E $\mathcal{P}$ if and only if f corresponds to an extreme point of $\mathcal{M}^*$. The set of extreme points of $\mathcal{M}^*$ consists of the point masses. Therefore E $\mathcal{P}$ is given by (8.16).

In the remainder of this chapter, F_α denotes the function $F_\alpha(z) = \dfrac{1}{(1-z)^\alpha}$ ($|z| < 1$) where $\alpha > 0$.

Lemma 8.10 *Suppose that* $\alpha > 0$, $\beta > 0$, *f is subordinate to* F_α *and g is subordinate to* F_β. Then $f \cdot g$ *is subordinate to* $F_{\alpha+\beta}$.

Proof: A function f is subordinate to F_α if and only if f is analytic in $\mathcal{D}$, $f(z) \neq 0$ for $|z| < 1$, Re $[f(z)^{1/\alpha}] > \frac{1}{2}$ for $|z| < 1$ and $f(0) = 1$. Hence the lemma follows if we show that the inequalities Re $[w_1^{1/\alpha}] > \frac{1}{2}$ and

$\text{Re}\,[w_2^{\,1/\beta}] > ½$ imply $\text{Re}\,[(w_1 w_2)^{1/(\alpha+\beta)}] > ½$. Let $s_1 = w_1^{\,1/\alpha}$, $s_2 = w_2^{\,1/\beta}$ and $t = \dfrac{\alpha}{\alpha+\beta}$. Then $\left[w_1 w_2\right]^{1/(\alpha+\beta)} = s_1^t\, s_2^{1-t}$. Hence it suffices to show that if $\text{Re}\, s_1 > ½$, $\text{Re}\, s_2 > ½$ and $0 < t < 1$, then $\text{Re}[s_1^t\, s_2^{1-t}] > ½$.

Let $s_1 = r_1 e^{i\theta_1}$ and $s_2 = r_2 e^{i\theta_2}$ where $|\theta_1| < \pi/2$, $|\theta_2| < \pi/2$, $r_1 > 0$ and $r_2 > 0$. Then $r_1 \cos\theta_1 > ½$ and $r_2 \cos\theta_2 > ½$. Hence

$$\text{Re}\left[s_1^t\, s_2^{1-t}\right] = r_1^t\, r_2^{1-t}\, \cos\left[t\theta_1 + (1-t)\theta_2\right] > \frac{\cos[t\theta_1 + (1-t)\theta_2]}{2(\cos\theta_1)^t\, (\cos\theta_2)^{1-t}}.$$

Thus it suffices to show that $\cos[t\theta_1 + (1-t)\theta_2] \geq (\cos\theta_1)^t\, (\cos\theta_2)^{1-t}$. This inequality follows from the fact that the function g defined by $g(\theta) = \log(\cos\theta)$ for $|\theta| < \pi/2$ is concave downward.

Suppose that the function F is analytic in $\mathcal{D}$, and let $\mathcal{F}$ denote the set of functions that are subordinate to F in $\mathcal{D}$. Schwarz's lemma implies that if f 0 $\mathcal{F}$ and $0 < r < 1$, then $\max\limits_{|z| \leq r} |f(z)| \leq \max\limits_{|z| \leq r} |F(z)|$. Hence $\mathcal{F}$ is locally bounded. The family of functions satisfying Schwarz's lemma is closed in H and hence $\mathcal{F}$ is closed in H. These properties of $\mathcal{F}$ imply that H$\mathcal{F}$ is compact. Moreover, EH$\mathcal{F}$ δ $\mathcal{F}$ by the Krein-Milman theorem.

Theorem 8.11 *Let* $\mathcal{G}_\alpha$ *denote the set of functions that are subordinate to* F_α *in* $\mathcal{D}$. *If* $\alpha \geq 1$, *then* H$\mathcal{G}_\alpha$ *consists of those functions f given by*

$$f(z) = \int_T \frac{1}{(1-\overline{\zeta}z)^\alpha}\, d\mu(\zeta) \qquad (|z| < 1) \qquad (8.18)$$

where μ 0 $\mathcal{M}^*$.

Proof: First suppose that $\alpha = 1$. For each ζ with $|\zeta| = 1$, the function $f(z) = \dfrac{1}{1-\overline{\zeta}z}$ belongs to $\mathcal{G}_1$. Thus each function given by (8.18), where μ 0 $\mathcal{M}^*$ and $\alpha = 1$, belongs to H$\mathcal{G}_1$. Next note that if f 0 EH$\mathcal{G}_1$, then f is analytic in $\mathcal{D}$, $\text{Re}\, f(z) > ½$ and $f(0) = 1$. Theorem 1.1 implies that f is of the form (8.18) with μ0 $\mathcal{M}^*$.

In the case $\alpha > 1$, it suffices to show that all functions in H$\mathcal{G}_\alpha$ are given by (8.18), where μ 0 $\mathcal{M}^*$. Suppose that f 0 EH$\mathcal{G}_\alpha$. Then f 0 $\mathcal{G}_\alpha$ and hence $f = g^\alpha$ where g 0 $\mathcal{G}_1$. We claim that g 0 E$\mathcal{G}_1$. To the contrary, suppose that $g = tg_1 + (1-t)g_2$ where g_1 0 $\mathcal{G}_1$, g_2 0 $\mathcal{G}_1$, $g_1 \neq g_2$ and $0 < t < 1$. Then

$$f = g \cdot g^{\alpha-1} = tg_1 \cdot g^{\alpha-1} + (1-t)g_2 \cdot g^{\alpha-1}.$$

By Lemma 8.10 this yields $f = t f_1 + (1-t) f_2$ where $f_1 \in \mathcal{G}_\alpha$ and $f_2 \in \mathcal{G}_\alpha$. Since $g_1 \neq g_2$ and $g \neq 0$ it follows that $f_1 \neq f_2$. Thus $f \notin \mathrm{E}\mathcal{G}_\alpha$ and hence $f \notin \mathrm{EH}\mathcal{G}_\alpha$. This contradiction verifies the claim.

Lemma 8.9 implies that $\mathrm{E}\mathcal{G}_1 = \left\{ g : g(z) = \dfrac{1}{1-\overline{\zeta}z}, |\zeta| = 1 \right\}$. Hence functions in $\mathrm{EH}\mathcal{G}_\alpha$ have the form $f(z) = \dfrac{1}{(1-\overline{\zeta}z)^\alpha}$ where $|\zeta| = 1$. Thus the Krein-Milman theorem implies that each function in $\mathrm{H}\mathcal{G}_\alpha$ is given by (8.18) where $\mu \in \mathcal{M}^*$.

Theorem 8.12 *If* $\varphi : \mathbb{D} \to \mathbb{D}$ *is analytic and* $\alpha \geq 1$, then φ *induces a composition operator on* $\mathcal{F}_\alpha$.

Proof: We must show that if $\alpha \geq 1$ and $f \in \mathcal{F}_\alpha$, then $g = f \circ \varphi \in \mathcal{F}_\alpha$. There exists $\mu \in \mathcal{M}$ such that (1.1) holds, and thus

$$g(z) = \int_T \frac{1}{[1-\overline{\zeta}\varphi(z)]^\alpha} \, d\mu(\zeta) \qquad (|z| < 1). \tag{8.19}$$

The Jordan decomposition theorem implies that we may assume that $\mu \in \mathcal{M}^*$.

First suppose that $\varphi(0) = 0$. Let $\mathcal{G}_\alpha$ be defined as in Theorem 8.11. Since $\mu \in \mathcal{M}^*$, the integral in (8.19) is a limit of convex combinations of functions in $\mathcal{G}_\alpha$ and hence $g \in \mathrm{H}\mathcal{G}_\alpha$. Hence Theorem 8.11 yields $g(z) = \displaystyle\int_T \frac{1}{(1-\overline{\zeta}z)^\alpha} \, d\upsilon(\zeta)$ for $|z| < 1$, where $\upsilon \in \mathcal{M}^*$. This proves the theorem in the case $\varphi(0) = 0$.

For the general case, let $b = \varphi(0)$ and let $h(z) = \dfrac{z + b}{1 + \overline{b}z}$ $(|z| < 1)$. By Theorem 8.1, $f \circ h \in \mathcal{F}_\alpha$ for every $f \in \mathcal{F}_\alpha$. Let $\omega(z) = \dfrac{\varphi(z) - b}{1 - \overline{b}\varphi(z)}$ for $|z| < 1$. Then $\omega: \mathbb{D} \to \mathbb{D}$ is analytic and $\omega(0) = 0$. The previous argument yields $f \circ h \circ \omega \in \mathcal{F}_\alpha$ for every $f \in \mathcal{F}_\alpha$. Since $h \circ \omega = \varphi$, the proof is complete.

In the case $0 < \alpha < 1$, not every analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ induces a composition operator on $\mathcal{F}_\alpha$. To see this, note that the function $f(z) = z$ belongs to $\mathcal{F}_\alpha$ for all α with $\alpha > 0$. It follows that if C_φ maps $\mathcal{F}_\alpha$ into $\mathcal{F}_\alpha$, then

$C_\varphi(f) = \varphi \in \mathcal{F}_\alpha$. Let $\varphi(z) = \varepsilon \sum_{n=1}^{\infty} \frac{1}{n^2} z^{2^n}$ ($|z| < 1$), where $\varepsilon > 0$ is chosen so that $\varepsilon \sum_{n=1}^{\infty} 1/n^2 < 1$. Then φ is analytic in $\mathcal{D}$ and $\varphi(D) \subset D$. However, as shown on p. 34, $\varphi \notin \mathcal{F}_\alpha$ when $0 < \alpha < 1$.

If φ induces a composition operator on $\mathcal{F}_\alpha$, then the operator norm of C_φ is denoted $\| C_\varphi \|_\alpha$ and is defined by

$$\| C_\varphi \|_\alpha = \sup_{\substack{f \in \mathcal{F}_\alpha \\ f \neq 0}} \frac{\| C_\varphi(f) \|_{\mathcal{F}_\alpha}}{\| f \|_{\mathcal{F}_\alpha}}.$$

An examination of the arguments yielding Theorem 8.12 shows that for $\alpha \geq 1$ and for any analytic function $\varphi : \mathcal{D} \to \mathcal{D}$, $\| C_\varphi \|_\alpha \leq \frac{A}{(1 - |\varphi(0)|)^\alpha}$ where the positive constant A depends only on α.

Theorem 8.12 yields a simple geometric condition sufficient to imply membership in $\mathcal{F}_\alpha$ when $\alpha \geq 1$. The argument uses the following lemma.

Lemma 8.13 *Let* N *be a positive integer. Suppose that* $|\zeta_n| = 1$ *and* $\alpha_n > 0$ *for* $n = 1, 2, \ldots, N$ *and* $\zeta_n \neq \zeta_m$ *for* $n \neq m$. *Suppose that the function g is analytic in a neighborhood of* $\mathcal{D}$. *Let*

$$f(z) = g(z) \Big/ \prod_{n=1}^{N} (z - \zeta_n)^{\alpha_n} \qquad (|z| < 1)$$

and let $\alpha = \max\{\alpha_n : 1 \leq n \leq N\}$. *Then* $f \in \mathcal{F}_\alpha$.

Proof: We give the proof in the case N = 2. The argument is similar for other values of N.

Suppose that $|\zeta| = |\sigma| = 1$, $\zeta \neq \sigma$, $\beta > 0$ and $\gamma > 0$. Suppose that g is analytic in a neighborhood of $\mathcal{D}$ and let

$$f(z) = \frac{g(z)}{(z - \zeta)^\beta (z - \sigma)^\gamma} \qquad (|z| < 1).$$

We shall show that $f \in \mathcal{F}_\alpha$ where $\alpha = \max\{\beta, \gamma\}$.

The function $z \mapsto g(z) / (z - \sigma)^\gamma$ is analytic at ζ, and hence

$$\frac{g(z)}{(z-\sigma)^{\gamma}} = \sum_{m=0}^{\infty} a_m \ (z-\zeta)^m$$

for z in some neighborhood of ζ. Let p be the least integer such that $p \geq \beta$ and let $s = p-\beta$. Then

$$f(z) = \sum_{m=0}^{p-1} \frac{a_m}{(z-\zeta)^{\beta-m}} + (z-\zeta)^s \ h(z) \tag{8.20}$$

where the function *h* is analytic in some neighborhood of ζ. Suppose that β is not an integer. Then for z 0 $\overline{D} \setminus \{\zeta\}$ with $z \neq \zeta$,

$$\frac{d}{dz}\left[(z-\zeta)^s \ h(z)\right] = (z-\zeta)^s \ h'(z) + s(z-\zeta)^{s-1} \ h(z).$$

Since $(z-\zeta)^s$ is bounded in $\overline{D} \setminus \{\zeta\}$ this implies that there is a constant A such that

$$\left|\frac{d}{dz}\left[(z-\zeta)^s \ h(z)\right]\right| \leq A \ |z-\zeta|^{s-1} \tag{8.21}$$

for z 0 $\overline{D}$, z near ζ, $z \neq \zeta$. Likewise if γ is not an integer, q is the least integer such that $q \geq \gamma$ and $t = q-\gamma$, then

$$f(z) = \sum_{m=0}^{q-1} \frac{b_m}{(z-\sigma)^{\gamma-m}} + (z-\sigma)^t \ k(z) \tag{8.22}$$

where k is an analytic function in some neighborhood of σ and b_m ($m = 0, 1, \ldots, q-1$) are suitable constants. We have

$$\left|\frac{d}{dz}\left[(z-\sigma)^t \ k(z)\right]\right| \leq B \ |z-\sigma|^{t-1} \tag{8.23}$$

for z 0 $\overline{D}$, z near σ, and $z \neq \sigma$, where B is a positive constant.

For z 0 $\overline{D} \setminus \{\zeta, \sigma\}$ let

$$r(z) = f(z) - \sum_{m=0}^{p-1} \frac{a_m}{(z-\zeta)^{\beta-m}} - \sum_{m=0}^{q-1} \frac{b_m}{(z-\sigma)^{\gamma-m}}. \tag{8.24}$$

Because of (8.20), (8.21) and (8.24) there is a constant C such that

$$|r'(z)| \le C\,|z-\zeta|^{s-1} \tag{8.25}$$

for $z \in \overline{D}$, z near ζ and $z \ne \zeta$. Likewise (8.22), (8.23) and (8.24) imply that

$$|r'(z)| \le D\,|z-\sigma|^{t-1} \tag{8.26}$$

for $z \in \overline{D}$, z near σ and $z \ne \sigma$, where D is a constant.

The function $z \mapsto (z-\tau)^u$ belongs to H^1 when $|\tau| = 1$ and $u > -1$. Hence the inequalities (8.25) and (8.26) and the fact that r' is analytic in $\overline{D} \setminus \{\zeta, \sigma\}$ imply that $r' \in H^1$ when β and γ are not integers. A similar argument shows that $r' \in H^1$ when only one of the numbers β and γ is not an integer. If both β and γ are integers, then $r = 0$. Therefore in general $r' \in H^1$. Since $H^1 \subset \mathcal{F}_1$ and since $\mathcal{F}_1 \subset \mathcal{F}_\alpha$ for $\alpha \ge 1$, Theorem 2.8 implies that $r \in \mathcal{F}_\delta$ for every $\delta > 0$.

Equation (8.24) gives

$$f = f_1 + f_2 + r \tag{8.27}$$

where $f_1(z) = \sum_{m=0}^{p-1} \frac{a_m}{(z-\zeta)^{\beta-m}}$ and $f_2(z) = \sum_{m=0}^{q-1} \frac{b_m}{(z-\sigma)^{\gamma-m}}$. The function $z \mapsto \frac{1}{(z-\zeta)^{\delta}}$ belongs to $\mathcal{F}_\beta$ when $0 < \delta \le \beta$ and hence $f_1 \in \mathcal{F}_\beta$. Likewise $f_2 \in \mathcal{F}_\gamma$. Theorem 2.10 implies that $f_1 \in \mathcal{F}_\alpha$ and $f_2 \in \mathcal{F}_\alpha$. Since $r \in \mathcal{F}_\alpha$ the relation (8.27) implies that $f \in \mathcal{F}_\alpha$. □

Theorem 8.14 *Suppose that $f\colon \mathbb{D} \to \mathbb{C}$ is analytic and let $\Phi = \mathbb{C} \setminus f(\mathbb{D})$.*

1) Suppose that Φ contains two rays. Let $\alpha\pi$ and $\beta\pi$ denote the angles at ∞ between these two rays where $\alpha \ge \beta$. If $\alpha < 2$ *then* $f \in \mathcal{F}_\alpha$.

2) If Φ contains a ray then $f \in \mathcal{F}_2$.

Proof: First suppose that Φ contains two rays. Let the angles at ∞ between these two rays be $\alpha\pi$ and $\beta\pi$, where $\alpha \ge \beta$. We may assume that the rays do not intersect. Let F denote a conformal mapping of $\mathbb{D}$ onto the complement of the rays. The Schwarz-Christoffel formula gives

$$F(z) = a \int_0^z \frac{(w-\zeta_1)(w-\zeta_2)}{(w-\zeta_3)^{\alpha+1}(w-\zeta_4)^{\beta+1}}\, dw + b \qquad (|z| < 1)$$

where a and b are complex numbers and ζ_1, ζ_2, ζ_3 and ζ_4 are distinct points on $\partial\mathbb{D}$. Thus

$$F'(z) = \frac{G(z)}{(z-\zeta_3)^{\alpha+1}(z-\zeta_4)^{\beta+1}}$$

where $G(z) = a(z-\zeta_1)(z-\zeta_2)$. Lemma 8.13 implies that F' belongs to $\mathcal{F}_{\alpha+1}$. Theorem 2.8 yields $F \in \mathcal{F}_\alpha$.

To complete the proof of the first assertion, let $\varphi = F^{-1} \circ f$. Then φ is analytic in $\mathbb{D}$ and $\varphi : \mathbb{D} \to \mathbb{D}$. Since $\alpha + \beta = 2$ and $\beta \leq \alpha$, it follows that $\alpha \geq 1$. Since $F \in \mathcal{F}_\alpha$, Theorem 8.12 implies that $f = F \circ \varphi \in \mathcal{F}_\alpha$.

The second assertion can be proved in a similar way. Suppose that Φ contains a ray. The conformal mapping of $\mathbb{D}$ onto the complement of a ray has the form $F(z) = \dfrac{P(z)}{(z-\zeta)^2}$ where P is a quadratic polynomial and $|\zeta| = 1$. This yields $F \in \mathcal{F}_2$ and hence Theorem 8.12 with $\alpha = 2$ gives $f \in \mathcal{F}_2$.

Next we show that if φ induces a composition operator on $\mathcal{F}_\alpha$ and $\beta > \alpha$, then φ induces a composition operator on $\mathcal{F}_\beta$. The following two lemmas are used in the proof.

Lemma 8.15 *Suppose that $|\zeta| = 1$, $0 < \alpha < \beta$, $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic and $\varphi(0) = 0$. Let*

$$f(z) = \frac{1}{[1-\overline{\zeta}\varphi(z)]^{\beta-\alpha}} \qquad (|z| < 1).$$

Then $f \cdot g \in \mathcal{F}_{\beta+1}$ for every $g \in \mathcal{F}_{\alpha+1}$ and there is a positive constant D *independent of ζ such that $\| f \cdot g \|_{\mathcal{F}_{\beta+1}} \leq D \| g \|_{\mathcal{F}_{\alpha+1}}$ for all $g \in \mathcal{F}_{\alpha+1}$.*

Proof: Fix $|\zeta| = 1$. Let $|\sigma| = 1$ and let

$$h(z) = f(z) \frac{1}{(1-\overline{\sigma}z)^{\alpha+1}} \qquad (|z| < 1).$$

Since $\varphi(0) = 0$, f is subordinate to $F_{\beta-\alpha}$. Lemma 8.10 implies that h is subordinate to $F_{\beta+1}$ for every $|\sigma| = 1$. By Theorem 8.11, h 0 $\mathcal{F}^*_{\beta+1}$ and hence $\| h \|_{\mathcal{F}_{\beta+1}} = 1$ for all $|\sigma| = 1$. An argument as in the proof of Theorem 6.5 shows that $f \cdot g$ 0 $\mathcal{F}_{\beta+1}$ for every g 0 $\mathcal{F}_{\alpha+1}$ and there is a positive constant D independent of ζ that $\| f \cdot g \|_{\mathcal{F}_{\beta+1}} \leq D \| g \|_{\mathcal{F}_{\alpha+1}}$ for all g 0 $\mathcal{F}_{\alpha+1}$ and for all $|\zeta| = 1$.

Lemma 8.16 *Suppose that the function* $\varphi : D \to D$ *is analytic and let* $\alpha > 0$. *The composition* C_φ *is a continuous linear operator on* $\mathcal{F}_\alpha$ *if and only if*

$$\sup \left\{ \left\| \frac{1}{[1-\overline{\zeta}\varphi(z)]^\alpha} \right\|_{\mathcal{F}_\alpha} : |\zeta| = 1 \right\} < \infty .$$

Proof: This is an easy argument similar to the proof given for Theorem 6.5.

Theorem 8.17 *Suppose that the function* $\varphi : D \to D$ *is analytic and* $0 < \alpha < \beta$. *If* φ *induces a composition operator on* $\mathcal{F}_\alpha$ *then* φ *induces a composition operator on* $\mathcal{F}_\beta$.

Proof: By Lemma 8.16, the hypotheses imply that there is a positive constant C such that

$$\left\| \frac{1}{[1-\overline{\zeta}\varphi(z)]^\alpha} \right\|_{\mathcal{F}_\alpha} \leq C$$

for all ζ with $|\zeta| = 1$. Theorem 2.8 implies that there is a positive constant A depending only on α such that

$$\left\| \frac{\alpha\overline{\zeta}\varphi'(z)}{[1-\overline{\zeta}\varphi(z)]^{\alpha+1}} \right\|_{\mathcal{F}_{\alpha+1}} \leq C\,A$$

for $|\zeta| = 1$.

Assume that $\varphi(0) = 0$. An application of Lemma 8.15 yields

$$\left\| \frac{\alpha\overline{\zeta}\varphi'(z)}{[1-\overline{\zeta}\varphi(z)]^{\beta+1}} \right\|_{\mathcal{F}_{\beta+1}} \leq D\,C\,A$$

for $|\zeta| = 1$ and thus

$$\left\| \frac{\beta\overline{\zeta}\varphi'(z)}{[1-\overline{\zeta}\varphi(z)]^{\beta+1}} \right\|_{\mathcal{F}_{\beta+1}} \leq \frac{\beta}{\alpha} DCA$$

for $|\zeta| = 1$. Hence Theorem 2.8 shows that $\dfrac{1}{[1-\overline{\zeta}\varphi(z)]^{\beta}} \in \mathcal{F}_\beta$ for $|\zeta| = 1$. Since $\varphi(0) = 0$, Theorem 2.8 gives

$$\left\| \frac{1}{[1-\overline{\zeta}\varphi(z)]^{\beta}} \right\|_{\mathcal{F}_{\beta}} \leq 1 + \frac{\beta}{\alpha} DCAB$$

for all $|\zeta| = 1$. Lemma 8.16 completes the proof in the case $\varphi(0) = 0$.

To prove the theorem in general, let $b = \varphi(0)$, $\psi(z) = \dfrac{z-b}{1-\overline{b}z}$ $(|z| < 1)$ and $\omega = \psi \circ \varphi$. Suppose that $f \in \mathcal{F}_\alpha$ and $0 < \alpha < \beta$. Theorem 8.1 yields $f \circ \psi \in \mathcal{F}_\alpha$. By assumption this implies that $(f \circ \psi) \circ \varphi \in \mathcal{F}_\alpha$, that is, $f \circ \omega \in \mathcal{F}_\alpha$ for every $f \in \mathcal{F}_\alpha$. Since $\omega(0) = 0$, the previous case implies that ω induces a composition operator on $\mathcal{F}_\beta$.

Suppose that $g \in \mathcal{F}_\beta$. Since ψ^{-1} is a conformal automorphism of $\mathbb{D}$, Theorem 8.1 yields $h = g \circ \psi^{-1} \in \mathcal{F}_\beta$. It follows that $h \circ \omega \in \mathcal{F}_\beta$. Since $h \circ \omega = g \circ \varphi$, the argument shows that $g \circ \varphi \in \mathcal{F}_\beta$ for every $g \in \mathcal{F}_\beta$.

Theorem 8.18 *Suppose that the function* $\varphi : \mathbb{D} \to \mathbb{D}$ *is analytic,* $\varphi(z) = \sum_{n=0}^{\infty} b_n z^n$ *for* $|z| < 1$ *and* $\sum_{n=1}^{\infty} n|b_n| < \infty$. *Then* φ *induces a composition operator on* $\mathcal{F}_\alpha$ *for every* $\alpha > 0$.

Proof: Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic and let $\alpha > 0$. Assume that $f \in \mathcal{F}_\alpha$ and let $g = f \circ \varphi$. Theorem 2.8 yields $f' \in \mathcal{F}_{\alpha+1}$ and hence Theorem 8.12 implies that $f' \circ \varphi \in \mathcal{F}_{\alpha+1}$. If $\varphi(z) = \sum_{n=0}^{\infty} b_n z^n$ then the assumption $\sum_{n=1}^{\infty} n|b_n| < \infty$ implies that φ' satisfies the hypotheses of Theorem 7.15.

Therefore $\varphi' \in \mathcal{M}_{\alpha+1}$. We have $g' = (f' \circ \varphi)\varphi'$ where $f' \circ \varphi \in \mathcal{F}_{\alpha+1}$ and $\varphi' \in \mathcal{M}_{\alpha+1}$. Thus $g' \in \mathcal{F}_{\alpha+1}$ and Theorem 2.8 yields $g \in \mathcal{F}_\alpha$.

Suppose that $\varphi : \mathcal{D} \to \mathcal{D}$ is analytic and $\varphi'' \in H^1$. Let

$$\varphi(z) = \sum_{n=0}^{\infty} b_n z^n \qquad (|z| < 1).$$

Hardy's inequality asserts that if $f \in H^1$ and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < 1),$$

then

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leq \pi \, \|f\|_{H^1} .$$

Since $\varphi'' \in H^1$, it follows that $\sum_{n=0}^{\infty} (n+2)\,|b_{n+2}| \leq \pi \, \|\varphi''\|_{H^1} < \infty$. By Theorem 8.18, φ induces a composition operator on $\mathcal{F}_\alpha$ for every $\alpha > 0$.

Corollary 8.19 *Each finite Blaschke product induces a composition operator on* $\mathcal{F}_\alpha$ *for every* $\alpha > 0$.

Little is known about which analytic functions $\varphi : \mathcal{D} \to \mathcal{D}$ induce a composition operator on $\mathcal{F}_\alpha$ for a given α where $0 < \alpha < 1$. In particular, this problem is unresolved for inner functions. One fact in this direction is that the function S defined by $S(z) = \exp\left[-\frac{1+z}{1-z}\right]$ $(|z| < 1)$ does not induce a composition operator on $\mathcal{F}_\alpha$ when $0 < \alpha \leq ½$. As noted previously, the condition $\varphi \in \mathcal{F}_\alpha$ is necessary for φ to induce a composition operator on $\mathcal{F}_\alpha$. In Chapters 2 and 3 we showed that $S \notin \mathcal{F}_\alpha$ for $0 < \alpha \leq ½$. In Chapter 9 we consider univalent maps $\varphi : \mathcal{D} \to \mathcal{D}$. We show that under certain conditions, such maps induce composition operators on $\mathcal{F}_\alpha$ for certain values of $\alpha < 1$.

NOTES

Theorem 1 was obtained by Hibschweiler and MacGregor [1989]. Corollary 2 appears in Hibschweiler and MacGregor [1992]. That reference also includes the result that the family $\mathcal{F}_0$ is closed under composition with conformal automorphisms of $\mathcal{D}$. Hallenbeck and Samotij [1993] proved the analogous result for $\mathcal{M}_0$. Corollaries 4 and 6 appear in Hibschweiler and Nordgren [1996; see p. 643]. Theorem 7 and several other results about divisibility and $\mathcal{F}_1$ are in Vinogradov, Goluzina and Havin [1970]. Holland [1973] characterized E$\mathcal{P}$ with an argument which does not depend on the Riesz-Herglotz formula. Because of the Krein-Milman theorem, Holland's result implies the Riesz-Herglotz formula. Theorem 11 was proved for $\alpha > 1$ by Brannan, Clunie and Kirwan [1973]. The case $\alpha = 1$ of Theorem 12 was shown by Bourdon and Cima [1988]. Hibschweiler and MacGregor [1989] proved Theorem 12 for $\alpha > 1$. Bourdon and Cima proved the case of Theorem 14 where $\alpha = \beta$. The general result is due to Hibschweiler and MacGregor [2004]. The Schwarz-Christoffel formula is in Goluzin [1969; see p. 77]. Hibschweiler [1998] proved Theorem 17. Additional results about composition operators on $\mathcal{F}_1$ are in Cima and Matheson [1998]. A survey of results about composition operators on other spaces of analytic functions is given by Cowen and MacCluer [1995] and by J.H. Shapiro [1993].

CHAPTER 9

Univalent Functions

Preamble. The significance of univalent functions in the study of the families $\mathcal{F}_\alpha$ was demonstrated in previous chapters, for example, in the introduction of suitable conformal mappings and especially in the context of subordination. Now we obtain further relations between univalent functions and the families $\mathcal{F}_\alpha$. The emphasis is on determining the values of α for which a univalent function or a class of univalent functions belongs to $\mathcal{F}_\alpha$.

The initial research about univalent functions and integral representations involving measures concerned questions about the extreme points and the convex hulls of various families. The measures which occur in such considerations are probability measures. Two results of this type are given in Theorem 9.1, which concerns starlike and convex mappings.

It is shown that each function which is analytic and univalent in $\mathcal{D}$ belongs to $\mathcal{F}_\alpha$ for $\alpha > 2$. Also, various univalent functions belong to $\mathcal{F}_2$. In particular, this occurs when the maximum modulus of such a function is somewhat restricted. We give examples of analytic univalent functions which do not belong to $\mathcal{F}_2$. It is an open problem to describe the linear span of the set of analytic univalent functions.

Additional facts are given about membership of a univalent function f in $\mathcal{F}_\alpha$ in terms of the maximum modulus of f. Also it is shown that if f is analytic and univalent in $\mathcal{D}$ and if $\sup_{|z|<1} |f(z)| < 1$, then f induces a composition operator on $\mathcal{F}_\alpha$ for all $\alpha > \beta_0$, where β_0 is a specific positive number.

Let $\mathcal{U}$ denote the set of functions that are analytic and univalent in $\mathcal{D}$ and let $\mathcal{S}$ denote the subset of $\mathcal{U}$ consisting of functions f normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}^*$ denote the class of functions $f \in \mathcal{S}$ for which $f(\mathcal{D})$ is starlike with respect to the origin. Also let $\mathcal{K}$ denote the subset of $\mathcal{S}$ for which $f(\mathcal{D})$ is convex. The families $\mathcal{S}$, $\mathcal{S}^*$ and $\mathcal{K}$ are compact subsets of H. These families have been studied extensively.

The family $\mathcal{S}^*$ is characterized as the set of functions f analytic in $\mathbb{D}$ such that $f(0) = 0$, $f'(0) = 1$ and

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0. \qquad (|z| < 1). \qquad (9.1)$$

Let $p(z) = zf'(z)/f(z)$ for $0 < |z| < 1$ and let $p(0) = 1$. Since $f(z) \neq 0$ for $0 < |z| < 1$, it follows that p is analytic in $\{z: 0 < |z| < 1\}$. The normalizations on f imply that p is analytic at 0. Therefore, $f \in \mathcal{S}^*$ if and only if $p \in \mathcal{P}$.

The family $\mathcal{K}$ is characterized as the set of functions f analytic in $\mathbb{D}$ such that $f(0) = 0$, $f'(0) = 1$ and

$$\text{Re}\left\{\frac{z\,f''(z)}{f'(z)} + 1\right\} > 0 \qquad (|z| < 1). \qquad (9.2)$$

Since $f'(z) \neq 0$ for $|z| < 1$ and $f'(0) = 1$, condition (9.2) is equivalent to the condition $p \in \mathcal{P}$, where $p(z) = \left[z\,f''(z)/f'(z)\right] + 1$ for $|z| < 1$.

Theorem 9.1 *The closed convex hull of $\mathcal{S}^*$ consists of functions given by*

$$f(z) = \int_T \frac{z}{(1-\overline{\zeta}z)^2}\,d\mu(\zeta) \qquad (|z| < 1) \qquad (9.3)$$

where $\mu \in \mathcal{M}^*$. The closed convex hull of $\mathcal{K}$ consists of functions given by

$$f(z) = \int_T \frac{z}{1-\overline{\zeta}z}\,d\mu(\zeta) \qquad (|z| < 1) \qquad (9.4)$$

where $\mu \in \mathcal{M}^*$.

Proof: Suppose that $f \in \mathcal{S}^*$. Let $p(z) = zf'(z)/f(z)$ for $0 < |z| < 1$ and let $p(0) = 1$. Then $p \in \mathcal{P}$ and hence Theorem 1.1 implies that

$$p(z) = \int_T \frac{1+\overline{\zeta}z}{1-\overline{\zeta}z}\,d\upsilon(\zeta) \qquad (|z| < 1) \qquad (9.5)$$

where υ 0 $\mathcal{M}^*$. If we let $g(z) = f(z)/z$ for $0 < |z| < 1$ and let $g(0) = 1$, then g is analytic in $\mathcal{D}$ and $g(z) \neq 0$ for $|z| < 1$. Hence $\log g$ is a well defined analytic function where $\log 1 = 0$, and

$$\frac{d}{dz}\left\{\log\ g(z)\right\} = \frac{\dfrac{zf'(z)}{f(z)} - 1}{z} = \frac{p(z) - 1}{z}.$$

Hence (9.5) and the condition $p(0) = 1$ yield

$$\begin{aligned}
\log\ g(z) &= \int_0^z \frac{p(w) - 1}{w}\, dw \\
&= \int_0^z \frac{1}{w}\left\{\int_T \frac{1 + \bar{\zeta}w}{1 - \bar{\zeta}w}\, d\upsilon(\zeta) - \int_T d\upsilon(\zeta)\right\} dw \\
&= \int_0^z \left\{\int_T \frac{2\bar{\zeta}}{1 - \bar{\zeta}w}\, d\upsilon(\zeta)\right\} dw \\
&= \int_T \left\{\int_0^z \frac{2\bar{\zeta}}{1 - \bar{\zeta}w}\, dw\right\} d\upsilon(\zeta) \\
&= \int_T -2\log(1 - \bar{\zeta}z)\, d\upsilon(\zeta).
\end{aligned}$$

Therefore

$$f(z) = z \exp\left\{\int_T -2\ \log\ (1 - \bar{\zeta}z)\ d\upsilon(\zeta)\right\} \qquad (|z| < 1). \tag{9.6}$$

Since υ 0 $\mathcal{M}^*$, υ is the weak* limit of finite convex combinations of point masses on T. Thus (9.6) implies that f is the limit of functions h having the form

$$h(z) = z \prod_{j=1}^{n} \frac{1}{(1 - \bar{\zeta}_j z)^{2\upsilon_j}} \tag{9.7}$$

where $|\zeta_j| = 1$, $\upsilon_j \geq 0$, $\sum_{j=1}^{n} \upsilon_j = 1$ and n = 1, 2, … . Since each function

$z \mapsto \dfrac{1}{(1 - \overline{\zeta}_j z)^{2\upsilon_j}}$ belongs to $\mathcal{F}^*_{2\upsilon_j}$ and $\sum_{j=1}^{n} \upsilon_j = 1$, Lemma 2.6 implies that

each function h in (9.7) has the form zk(z) and k 0 $\mathcal{F}_2^*$. Therefore f also has this form. This proves that (9.3) holds for each f 0 $\mathcal{S}^*$, where μ 0 $\mathcal{M}^*$.

Let $\mathcal{V}$ denote the set of functions defined by (9.3) where μ varies in $\mathcal{M}^*$. Since $\mathcal{V}$ is a closed convex set, the previous argument implies that H$\mathcal{S}^*$ δ $\mathcal{V}$. The definition of $\mathcal{V}$ shows that $\mathcal{V}$ is the closed convex hull of the set of functions

$z \mapsto \dfrac{z}{(1 - \overline{\zeta} z)^2}$ where ζ varies on T. Each such function belongs to $\mathcal{S}^*$ because it maps $\mathcal{D}$ one-to-one onto the complement of the ray

$$\{w = t\zeta : t \leq -1/4\}.$$

Therefore $\mathcal{V}$ δ H$\mathcal{S}^*$. This proves that H$\mathcal{S}^*$ = $\mathcal{V}$, which is the first assertion in this theorem.

The characterizations of $\mathcal{S}^*$ and $\mathcal{K}$ described earlier in (9.1) and (9.2) imply that f 0 $\mathcal{K}$ if and only if g 0 $\mathcal{S}^*$ where $g(z) = zf'(z)$. This mapping $f \mapsto g$ from $\mathcal{K}$ into $\mathcal{S}^*$ is a linear homeomorphism, and f is obtained from g by

$$f(z) = \int_0^z \frac{g(w)}{w}\, dw.$$

Hence this mapping also gives a linear homeomorphism of H$\mathcal{K}$ onto H$\mathcal{S}^*$. If

$$g(z) = \int_T \frac{z}{(1 - \overline{\zeta} z)^2}\, d\mu(\zeta) \qquad (|z| < 1)$$

then

$$f(z) = \int_T \frac{z}{1 - \overline{\zeta} z}\, d\mu(\zeta).$$

Therefore the result about H$\mathcal{S}^*$ in the first part of the theorem yields the assertion about H$\mathcal{K}$.

Arguments similar to ones given in previous chapters (such as in the proof of Theorem 3.10) show that each function given by (9.3) where μ 0 $\mathcal{M}$ belongs to $\mathcal{F}_2$. Therefore $\mathcal{S}^*$ δ $\mathcal{F}_2$. Also, if $f(0) = 0$ and f 0 $\mathcal{F}_2$ then f can be represented by (9.3) with μ 0 $\mathcal{M}$. We also note that (9.4) implies that $\mathcal{K}$ δ $\mathcal{F}_1$.

The next theorem depends on the following lemma, which is called Prawitz's inequality.

Lemma 9.2 *Suppose that the function f is analytic and univalent in* $\mathcal{D}$ *and* $f(0) = 0$. *Let* $M(r) = \max_{|z|=r} |f(z)|$ *for* $0 < r < 1$. *If* $0 < p < \infty$ *then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta \le p \int_0^r \frac{M^p(s)}{s} \, ds \tag{9.8}$$

for $0 < r < 1$.

Theorem 9.3 *Suppose that f is analytic and univalent in* $\mathcal{D}$. *For* $0 < r < 1$ *let* $N(r) = (1-r)\,M(r)$. *If* N *is Lebesgue integrable on* (0, 1) *then f* 0 $\mathcal{F}_2$.

Proof: Suppose that f is analytic in $\mathcal{D}$ and M and N are as defined above. For $0 < r < 1$ let $P(r) = \int_0^r M(\rho)\, d\rho$. Since $M(r) \ge 0$ the integrability of N is equivalent to the integrability of P. This is a consequence of the relation

$$\int_0^r (1-\rho)\,M(\rho)\, d\rho = (1-r)\,P(r) + \int_0^r P(\rho)\, d\rho$$

and the fact that if P is integrable then $\lim_{r\to 1-} (1-r)\,P(r) = 0$.

To prove the theorem we may assume that $f(0) = 0$. For $|z| < 1$ let

$$g(z) = \int_0^z f(w)\, dw. \tag{9.9}$$

By Theorem 2.8, $f = g'$ 0 $\mathcal{F}_2$ if and only if g 0 $\mathcal{F}_1$. Thus it suffices to prove that g 0 H^1.

We have $g(z) = z \int_0^1 f(tz)\, dt$. Since f is univalent in $\mathcal{D}$, Lemma 9.2 applies where p = 1. Hence

$$\int_{-\pi}^{\pi} |g(re^{i\theta})|\, d\theta \le \int_0^1 \int_{-\pi}^{\pi} |f(tre^{i\theta})|\, d\theta\, dt$$

$$\le \int_0^1 \{2\pi \int_0^{tr} \frac{M(s)}{s}\, ds\}\, dt$$

$$\le 2\pi \int_0^1 Q(t)\, dt$$

where $Q(t) = \int_0^t \frac{M(s)}{s}\, ds$ for $0 < t < 1$. Since $P \in L^1((0, 1))$ and $M(0) = 0$, $Q \in L^1((0, 1))$. Therefore

$$\int_{-\pi}^{\pi} |g(re^{i\theta})|\, d\theta \le 2\pi \int_0^1 Q(t)\, dt < \infty$$

for $0 < r < 1$. This shows that $g \in H^1$.

If $f \in \mathcal{U}$ then $|f(z)| \le |f(0)| + |f'(0)| \frac{|z|}{(1-|z|)^2}$ for $|z| < 1$ and hence $M(r) \le \frac{A}{(1-r)^2}$ for some $A > 0$. Theorem 9.3 shows that if the function f is univalent and the growth of M is somewhat restricted then $f \in \mathcal{F}_2$.

As already noted, the family $\mathcal{S}^*$ of starlike functions is a subset of $\mathcal{F}_2$. We give further examples of families of univalent functions with this property. Let γ be a real number with $|\gamma| < \pi/2$. A function f is called γ-spirallike if f is analytic in $\mathcal{D}$, $f(0) = 0$, $f'(0) = 1$ and $\text{Re}\left\{e^{-i\gamma} \frac{z f'(z)}{f(z)}\right\} > 0$ for $|z| < 1$. The case $\gamma = 0$ defines the class $\mathcal{S}^*$. Each γ-spirallike function is univalent in $\mathcal{D}$. The ranges of such mappings are characterized as follows: if $w \neq 0$ and $w \in f(\mathcal{D})$ then the logarithmic spiral $\zeta = w \exp[-e^{-i\gamma}t]$, $-\infty < t < \infty$, lies in $f(\mathcal{D})$.

The function

$$f(z) = \frac{z}{(1-z)^{c}} \quad (|z| < 1) \tag{9.10}$$

where $c = 2e^{i\gamma}\cos\gamma$ is γ-spirallike as is easy to verify. If g is γ-spirallike then $\frac{g(z)}{z}$ is subordinate to $\frac{f(z)}{z}$. This implies that if $0 < r < 1$, then

$$\begin{aligned}
\max_{|z|=r} |g(z)| &\le \max_{|z|=r} |f(z)| \\
&= r \max_{|z|=r} \exp\left\{2\cos^2\gamma \log\frac{1}{|1-z|} - 2\cos\gamma\sin\gamma\arg\left(\frac{1}{1-z}\right)\right\} \\
&< \exp(\pi/2)\,\frac{1}{(1-r)^{\beta}}
\end{aligned}$$

where $\beta = 2\cos^2\gamma$. In particular, if $\gamma \ne 0$, g satisfies

$$\max_{|z|=r} |g(z)| \le \frac{A}{(1-r)^{\beta}} \tag{9.11}$$

where $A > 0$ and $\beta < 2$. Theorem 9.3 implies that $g \in \mathcal{F}_2$. This also holds when $\gamma = 0$ because of Theorem 9.1. Thus every spirallike function belongs to $\mathcal{F}_2$.

A second example is the family of close-to-convex mappings, which is denoted $\mathcal{C}$. A function f belongs to $\mathcal{C}$ provided that f is analytic in $\mathbb{D}$, $f(0) = 0$, $f'(0) = 1$ and there is a function $g \in \mathcal{K}$ and a complex number b such that

$$\operatorname{Re}\left\{b\,\frac{z f'(z)}{g(z)}\right\} > 0 \qquad (|z| < 1). \tag{9.12}$$

Each function in $\mathcal{C}$ is univalent in $\mathbb{D}$. The ranges of functions $f \in \mathcal{C}$ are characterized as follows: $\mathbb{C} \setminus f(\mathbb{D})$ is a union of closed rays which do not intersect except perhaps at the finite endpoints of the rays. Hence Theorem 8.14, part 2 implies that $f \in \mathcal{F}_2$.

Next we give examples of analytic univalent functions which do not belong to $\mathcal{F}_2$. The argument depends on the following lemma. The proof for this lemma is the same as the proof of Theorem 6.10.

Lemma 9.4 *Suppose that* f 0 $\mathcal{F}_\alpha$ *for some* $\alpha > 0$ *and let* $g(z) = (1-z)^\alpha f(z)$ *for* $|z| < 1$. *Then the curve* $w = g(r)$, $0 \leq r < 1$, *is rectifiable.*

Theorem 9.5 *There exist functions that are analytic and univalent in* $\mathcal{D}$ *and do not belong to* $\mathcal{F}_2$.

Proof: Let ψ be a differentiable real-valued function on $(-\infty, \infty)$ such that ψ' is bounded, $(\psi')^2$ 0 $L^1((-\infty, \infty))$ and $\lim\limits_{u\to\infty} \psi(u) = \infty$. For example, we may let $\psi(u) = u^\beta$ for $u \geq 1$, where $0 < \beta < ½$, and define $\psi(u)$ for $u < 1$ suitably to obey the various conditions. Let

$$\Omega = \{w = u + iv: \psi(u) < v < \psi(u) + \pi, -\infty < u < \infty\}.$$

Let w_0 0 Ω and let h denote the unique function that is analytic in Ω, maps Ω one-to-one onto $\{s: |\mathrm{Im}\, s| < \pi/2\}$ and satisfies $h(w_0) = 0$, with

$$\lim_{\substack{w\to-\infty \\ w\in\Omega}} \mathrm{Re}\, h(w) = -\infty$$

and

$$\lim_{\substack{w\to\infty \\ w\in\Omega}} \mathrm{Re}\, h(w) = \infty.$$

The function $z \mapsto s$ where $s(z) = \log \dfrac{1+z}{1-z}$ maps $\mathcal{D}$ one-to-one onto $\{s: |\mathrm{Im}\, s| < \pi/2\}$, and the function $w \mapsto \zeta$ where $\zeta(w) = \exp(2w)$ is univalent in Ω. Let f denote the function $z \mapsto \zeta$ defined as the composition of $z \mapsto s$, $s \mapsto w$ and $w \mapsto \zeta$. Then f is analytic and univalent in $\mathcal{D}$.

Let $a = \mathrm{Re}\, w_0$. For w 0 Ω with $\mathrm{Re}\, w > a$, let $b = \mathrm{Re}\, w$. By construction there is a constant $M > 0$ such that $|\psi'(u)| \leq M$ for $-\infty < u < \infty$. It follows that

$$\mathrm{Re}\, \{h(w) - h(w_0)\} < \int_a^b \left\{1 + (\psi'(u))^2\right\} du + 12\pi\,(1 + M^2). \tag{9.13}$$

This inequality appears in Evgrafov [1966; see p. 140]. Since $(\psi')^2$ is integrable on $(-\infty, \infty)$ this yields

$$\text{Re } h(w) < \text{Re } w + A \quad (9.14)$$

for $w \in \Omega$ with Re w > Re w_0, where A is a positive constant. If $|z| < 1$ and z is sufficiently near 1 then the corresponding numbers w satisfy Re w > Re w_0, and hence (9.14) yields

$$|f(z)| = \exp(2\,\mathrm{Re}\,w) > \exp(2\,\mathrm{Re}\,s - 2A) = \exp(-2A)\left|\frac{1+z}{1-z}\right|^2.$$

Therefore there is a positive constant B and a real number r_0 such that $0 < r_0 < 1$ and

$$|f(r)| \geq \frac{B}{(1-r)^2} \tag{9.15}$$

for $r_0 \leq r < 1$.

The image of $[r_0, 1)$ under the mapping $z \longmapsto w$ is connected and $\lim\limits_{u\to\infty} \psi(u) = \infty$. Hence there is a positive integer m and an increasing sequence $\{r_n\}$ $(n = 1,2,\ldots)$ such that $r_1 > r_0$, $\lim\limits_{n\to\infty} r_n = 1$ and

$$\mathrm{Im}\ w_n = \left(m + \frac{n-1}{2}\right)\pi \tag{9.16}$$

for $n = 1,2,\ldots$, where w_n denotes the number corresponding to r_n under the mapping $z \longmapsto w$. We have $\arg f(r_n) = 2m\pi + (n-1)\pi$ and thus $f(r_n) > 0$ when n is odd and $f(r_n) < 0$ when n is even.

For $|z| < 1$ let $g(z) = (1-z)^2 f(z)$. Then $g(r_n) > 0$ when n is odd and $g(r_n) < 0$ when n is even. Also (9.15) gives $|g(r_n)| \geq B$ for all n. Since $B > 0$ this shows that the curve $w = g(r)$, $0 \leq r < 1$, is not rectifiable. Lemma 9.4 with $\alpha = 2$ yields $f \notin \mathcal{F}_2$. □

The examples given to prove Theorem 9.5 and those given by (9.10) map $\mathbb{D}$ onto the complement of spirals. The spirals for the function in the theorem tend to ∞ turning at a rate significantly slower than the spirals associated with (9.10).

The problem of characterizing the analytic univalent functions which belong to $\mathcal{F}_2$ is unresolved. The theorem below gives some information about the measures that represent such functions. A reference for this result is given in the Notes.

Theorem 9.6 *Suppose that the function f is univalent in* $\mathbb{D}$ *and*

$$f(z) = \int_T \frac{1}{(1 - \overline{\zeta} z)^2} \, d\mu(\zeta) \qquad (|z| < 1)$$

where $\mu \; 0 \; \mathcal{M}$. *Then there is at most one atomic point for* μ *and the continuous component of* μ *is absolutely continuous.*

The next theorem complements Theorem 9.3. It shows that further restrictions on the maximum modulus yield additional information about membership in the families $\mathcal{F}_\alpha$.

Theorem 9.7 *Let* $\beta_0 = \frac{1}{2} - \frac{1}{320}$. *Suppose that the function f is analytic and univalent in* $\mathcal{D}$ *and*

$$|f(z)| \leq \frac{A}{(1 - |z|)^\beta} \qquad (|z| < 1) \qquad (9.17)$$

where A *is a positive constant and* $0 < \beta \leq 2$. *If* $\beta \geq \beta_0$ *then* $f \; 0 \; \mathcal{F}_\alpha$ *for every* $\alpha > \beta$.

Proof: Suppose that $\beta_0 \leq \beta \leq 2$ and the function f is analytic and univalent in $\mathcal{D}$ and satisfies (9.17). This implies that there is a positive constant B such that

$$\int_{-\pi}^{\pi} \left| f'(re^{i\theta}) \right| d\theta \leq \frac{B}{(1-r)^\beta} \qquad (9.18)$$

for $0 \leq r < 1$. A reference for this result is in the Notes. Hence if $\alpha > \beta$ then

$$\int_0^1 \int_{-\pi}^{\pi} \left| f'(re^{i\theta}) \right| (1-r)^{\alpha-1} \, d\theta \, dr \leq B \int_o^1 (1-r)^{\alpha-1-\beta} \, dr < \infty.$$

Theorem 2.14 implies $f \; 0 \; \mathcal{F}_\alpha$.

When $\beta = 2$ the condition (9.17) is not a restriction on f. Hence every analytic univalent function f belongs to $\mathcal{F}_\alpha$ for $\alpha > 2$. This can be proven without using (9.18), as follows. The relation (2.20) gives a one-to-one mapping $f \mapsto g$ of $\mathcal{F}_\alpha$ onto $\mathcal{F}_1$ when $\alpha > 1$. Since

$$|f(z)| \leq \frac{A}{(1-|z|)^2} \qquad (|z| < 1)$$

it follows that g 0 H^1 and hence g 0 $\mathcal{F}_1$ when $\alpha > 2$. Thus f 0 $\mathcal{F}_\alpha$ for all $\alpha > 2$.

Theorem 9.7 is not valid for small values of β. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(|z| < 1)$ and f 0 $\mathcal{F}_\alpha$, then $a_n = O(n^{\alpha-1})$, as was shown in Chapter 2. However, there are bounded univalent functions f for which $a_n \neq O(n^{-0.83})$.

Next we show that Theorem 9.7 is sharp.

Theorem 9.8 *For each real number* β *such that* $0 < \beta < 2$, *there is a function f which is analytic and univalent in* $\mathcal{D}$, *satisfies* (9.17) *and* $f \notin \mathcal{F}_\beta$.

Proof: Suppose that $0 < \beta < 2$ and let $\gamma = (2\beta - \beta^2)^{1/2}$. Then $\gamma > 0$. Let

$$f(z) = \frac{1}{(1-z)^{\beta+i\gamma}} \qquad (|z| < 1)\,. \tag{9.19}$$

Then f is analytic in $\mathcal{D}$. Since $|f(z)| \leq \dfrac{\exp(\gamma\, \pi/2)}{(1-|z|)^\beta}$, (9.17) holds. The function $z \longmapsto w$ where $w = \log \dfrac{1}{1-z}$ maps D one-to-one onto a domain contained in the strip $\{w\colon |\text{Im } w| < \pi/2\}$, and the function $w \longmapsto \zeta$ where $\zeta = (\beta + i\gamma)w$ maps this strip one-to-one onto the strip

$$\Phi = \left\{\zeta = u + iv : \left|v - \frac{\gamma}{\beta} u\right| < \frac{\pi}{2}\,\frac{\beta^2 + \gamma^2}{\beta}\right\}.$$

Each line parallel to the imaginary axis meets Φ in a line segment with length $\pi \dfrac{\beta^2 + \gamma^2}{\beta}$, which equals 2π. Hence the exponential function is univalent in Φ. Since

$$f(z) = \exp\left[(\beta + i\gamma) \log \frac{1}{1-z}\right],$$

we conclude that f is univalent in $\mathcal{D}$.

For $|z| < 1$ let $g(z) = (1-z)^\beta f(z)$. Then $g(z) = \dfrac{1}{(1-z)^{i\gamma}}$. Each closed interval in r contained in [0,1) and corresponding to an interval of length $2\pi/\gamma$ in $\log 1/(1-r)$ is mapped onto T by g. Thus g maps [0,1) onto T, covered infinitely often. Hence the curve $w = g(r)$, $0 \leq r < 1$, is not rectifiable. Lemma 9.4 shows that $f \notin \mathcal{F}_\beta$.

Lemma 9.9 *Let* $\beta_0 = \frac{1}{2} - \frac{1}{320}$. *Suppose that the function* f *is analytic, univalent and bounded in* $\mathbb{D}$. *If* $\alpha > \beta_0$ *then* f 0 $\mathcal{F}_\alpha$ *and there is a positive constant* A *depending only on* α *such that*

$$\| f \|_{\mathcal{F}_\alpha} \leq A \| f \|_{H^\infty} . \tag{9.20}$$

Proof: Suppose that f is analytic, univalent and bounded in $\mathbb{D}$. There is a positive constant B which does not depend on f such that

$$\int_{-\pi}^{\pi} \left| f'(re^{i\theta}) \right| d\theta \leq B \| f \|_{H^\infty} \frac{1}{(1-r)^{\beta_0}} \tag{9.21}$$

for $0 \leq r < 1$. A reference for this fact is given in the Notes. If $\alpha > \beta_0$ this yields

$$\int_0^1 \int_{-\pi}^{\pi} \left| f'(re^{i\theta}) \right| (1-r)^{\alpha-1} \, d\theta \, dr \leq B \| f \|_{H^\infty} \int_0^1 (1-r)^{\alpha-1-\beta_0} \, dr < \infty .$$

Thus Theorem 2.14 implies that f 0 $\mathcal{F}_\alpha$ when $\alpha > \beta_0$. Moreover, Theorem 2.14 and (9.21) yield (9.20) where A depends only on α.

Theorem 9.10 *Let* $\beta_0 = \frac{1}{2} - \frac{1}{320}$. *If the function* φ *is analytic and univalent in* $\mathbb{D}$ *and* $\|\varphi\|_{H^\infty} < 1$, *then* φ *induces a composition operator on* $\mathcal{F}_\alpha$ *for* $\alpha > \beta_0$.

Proof: Suppose that the function φ satisfies the hypotheses of the theorem. Because of Theorem 8.9 (or Theorem 8.12), we may assume that $\beta_0 < \alpha < 1$. The function $z \mapsto \dfrac{1}{(1-\overline{\zeta}z)^\alpha}$ is univalent in $\mathbb{D}$ for $|\zeta| = 1$ and $0 < \alpha \leq 2$. Since φ is univalent, the function $z \mapsto \dfrac{1}{[1-\overline{\zeta}\varphi(z)]^\alpha}$ is univalent in $\mathbb{D}$ for each ζ with $|\zeta| = 1$. Since $\| \varphi \|_{H^\infty} < 1$,

$$\left\| \frac{1}{[1-\bar{\zeta}\varphi(z)]^{\alpha}} \right\|_{H^\infty} \le \frac{1}{(1-\|\varphi\|_{H^\infty})^{\alpha}}$$

for $|\zeta| = 1$. Lemma 9.9 yields

$$\left\| \frac{1}{[1-\bar{\zeta}\varphi(z)]^{\alpha}} \right\|_{\mathcal{F}_\alpha} \le \frac{A}{(1-\|\varphi\|_{H^\infty})^{\alpha}}$$

for $|\zeta| = 1$. Hence the set of functions

$$\left\{ z \mapsto \frac{1}{[1-\bar{\zeta}\varphi(z)]^{\alpha}} : |\zeta| = 1 \right\}$$

is a norm-bounded subset of $\mathcal{F}_\alpha$. Lemma 8.16 implies that $g \circ \varphi \in \mathcal{F}_\alpha$ for every $g \in \mathcal{F}_\alpha$.

When $\alpha \geq \frac{1}{2}$ Lemma 9.9 holds more generally for functions f in the Dirichlet space $\mathcal{D}_1$. Recall that $f \in \mathcal{D}_1$ provided that f is analytic in $\mathcal{D}$ and $\sum_{n=1}^{\infty} n|a_n|^2 < \infty$ where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($|z| < 1$). If f is analytic, univalent and bounded in $\mathcal{D}$, then $f \in \mathcal{D}_1$ because the area of $f(\mathcal{D})$ is finite. To prove the more general result, assume that $\sum_{n=1}^{\infty} n\,|a_n|^2 < \infty$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($|z| < 1$). Let

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{A_n(\frac{1}{2})} z^n \qquad (|z| < 1).$$

The asymptotic estimate (2.9) yields $g \in H^2$. Since $H^2 \subset H^1 \subset \mathcal{F}_1$, $g \in \mathcal{F}_1$. The remarks after (1.18) show that $f \in \mathcal{F}_{\frac{1}{2}}$. By Theorem 2.10, $f \in \mathcal{F}_\alpha$ for every $\alpha \geq \frac{1}{2}$.

The conclusion of Theorem 9.10 is not valid for small α. As noted in the remarks after Theorem 9.7, there are bounded univalent functions f which do not belong to $\mathcal{F}_\alpha$ for $\alpha < .17$. Let $M > \|f\|_{H^\infty}$ and let $\varphi(z) = f(z)/M$. Then φ is univalent and $\|\varphi\|_{H^\infty} < 1$, but φ does not induce a composition operator on $\mathcal{F}_\alpha$ for $\alpha < .17$.

It would be interesting to complement Theorem 9.10 with geometric and analytic conditions on a univalent function $\varphi: \mathcal{D} \to \mathcal{D}$ which imply that φ induces a composition operator on $\mathcal{F}_\alpha$ when $\|\varphi\|_{H^\infty} = 1$ and $0 < \alpha < 1$.

NOTES

Theorem 1 was proved by Brickman, MacGregor and Wilken [1971]. Facts about extreme points and convex hulls of families of functions in the context of geometric function theory are contained in Duren [1983; see Chapter 9], Hallenbeck and MacGregor [1984], and Schober [1975]. A proof of the Prawitz inequality is in Duren [1983; see p. 61]. Theorems 3 and 5 are in MacGregor [1987]. For facts about spirallike and close-to-convex mappings see Duren [1983; Chapter 2]. The initial argument for Theorem 5 depended on the construction of a suitable functional. A simplification of that argument was obtained by Hibschweiler and MacGregor [1990], and is given in the text. Theorem 6 was proved by Bass [1990]. Theorems 7 and 8 are in Hibschweiler and MacGregor [1990]. The result expressed by (9.18) was proved by Baernstein [1986]. The inequality (9.21) for a bounded univalent function is in Pommerenke [1975; see p. 131]. Theorem 10 was proved by Hibschweiler [1998] for $\alpha \geq ½$ using the argument related to the Dirichlet space given in the text.

A Characterization of Cauchy Transforms

Preamble. The family $\mathcal{F}_1$ was defined as the set of functions represented by formula (1.1) for $|z| < 1$. The formula (1.1) is well defined for $z \in \mathbb{C} \setminus \mathrm{T}$ and it defines an analytic function on $\mathbb{C} \setminus \mathrm{T}$. If we define $f(\infty) = 0$ then f extends analytically to $\mathbb{C}^{\infty} \setminus \mathrm{T}$.

The main development in this chapter yields a characterization of the functions analytic in $\mathbb{C}^{\infty} \setminus \mathrm{T}$ which can be represented by such a Cauchy transform. The integral means of $|f|$ on circles interior to T and exterior to T form one component of the characterization. A second component deals with the integrability of an associated function which is defined on T as a limit of the combined behavior of f inside and outside T. The argument begins by obtaining certain properties of subharmonic functions, and continues with the construction of a specific subharmonic function in $\mathbb{C}$.

We consider the set of functions defined on $\mathbb{C} \setminus \mathrm{T}$ by

$$f(z) = \int_{\mathrm{T}} \frac{1}{1 - \overline{\zeta} z} \, d\mu(\zeta) \tag{10.1}$$

where $\mu \in \mathcal{M}$. We shall show that this set of functions is in one-to-one correspondence with the set of measures $\mathcal{M}$.

The formula (10.1) implies that the function f is analytic in $\mathbb{C} \setminus \mathrm{T}$. If $|z| < 1$, then

$$f(1/z) = z \int_{\mathrm{T}} \frac{1}{z - \overline{\zeta}} \, d\mu(\zeta).$$

This implies that if we define $f(\infty) = 0$, then f is extended analytically to the point at infinity.

Theorem 10.1 *The mapping* $\mu \mapsto f$ *given by* (10.1) *is one-to-one from* $\mathcal{M}$ *to the set of functions that are analytic in* $\mathbb{C}^{\infty} \setminus \mathrm{T}$ *and vanish at infinity.*

Proof: Suppose that (10.1) holds where $\mu \in \mathcal{M}$ and let $f(\infty) = 0$. Also suppose that

$$g(z) = \int_T \frac{1}{1-\overline{\zeta}z}\, d\upsilon(\zeta) \qquad (z \in \mathbb{C} \setminus T) \qquad (10.2)$$

where $\upsilon \in \mathcal{M}$, and let $g(\infty) = 0$. Let f_1 denote the restriction of f to D and let f_2 denote the restriction of f to $\mathbb{C}^\infty \setminus \overline{D}$. Let g_1 and g_2 be defined in the analogous way for the function g.

Assume that $f = g$. Then $f_1 = g_1$ and $f_2 = g_2$. Since the Taylor coefficients of f_1 and g_1 are equal,

$$\int_T \overline{\zeta}^n\, d\mu(\zeta) = \int_T \overline{\zeta}^n\, d\upsilon(\zeta) \qquad (10.3)$$

for n = 0, 1, … . If $|z| > 1$, then

$$\begin{aligned} f_2(z) &= \int_T \frac{1}{1-\overline{\zeta}z}\, d\mu(\zeta) \\ &= \int_T \frac{1}{-\overline{\zeta}z\,(1-1/(\overline{\zeta}z))}\, d\mu(\zeta) \\ &= \int_T \frac{-1}{\overline{\zeta}z} \sum_{n=0}^{\infty} \frac{1}{(\overline{\zeta}z)^n}\, d\mu(\zeta) \\ &= \sum_{n=0}^{\infty} \left(\int_T \frac{-1}{\overline{\zeta}^{n+1}}\, d\mu(\zeta) \right) \frac{1}{z^{n+1}} \\ &= \sum_{n=1}^{\infty} \left(-\int_T \zeta^n\, d\mu(\zeta) \right) \frac{1}{z^n}. \end{aligned}$$

Likewise,

$$g_2(z) = \sum_{n=1}^{\infty} \left(-\int_T \zeta^n\, d\upsilon(\zeta) \right) \frac{1}{z^n} \qquad (|z| > 1).$$

Since $f_2 = g_2$ and since the Laurent expansion is unique, we conclude that

$$\int_T \zeta^n \, d\mu(\zeta) = \int_T \zeta^n \, d\upsilon(\zeta)$$

for n = 1, 2, … . When combined with (10.3), this yields

$$\int_T \overline{\zeta}^n \, d\mu(\zeta) = \int_T \overline{\zeta}^n \, d\upsilon(\zeta) \tag{10.4}$$

for all integers n. If $\lambda = \mu - \upsilon$ then Theorem 1.2 implies that $\lambda = 0$. Hence $\mu = \upsilon$ and the mapping is one-to-one.

Let $p > 0$ and let $D' = \{z \in \mathbb{C}^\infty : |z| > 1\}$. Then $H^p(D')$ is defined as the set of functions f that are analytic in D' and satisfy $\sup_{r>1} M_p(r, f) < \infty$ where

$$M_p(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta \qquad (r > 1).$$

Facts about $H^p(D')$ generally follow from the corresponding facts about H^p through the change of variables $z \mapsto 1/z$. For example, this can be used to prove that if $f \in H^p(D')$ and

$$\| f \|_{H^p(D')} \equiv \sup_{r>1} M_p(r, f)$$

then $\lim_{r\to 1+} M_p(r, f) = \| f \|_{H^p(D')}$. For simplicity we write $\| f \|_p$ instead of $\| f \|_{H^p(D')}$. We continue to use the same notation $\| f \|_p$ for the norm in H^p. The meaning will be clear from the context.

Suppose that f is given by (10.1) for $z \in \mathbb{C} \setminus T$, where $\mu \in \mathcal{M}$. Let f_1 and f_2 denote the restrictions of f to $\mathcal{D}$ and $\mathbb{C} \setminus \overline{D}$ respectively. We shall derive a number of properties of f_1 and f_2. These results will be summarized in Theorem 10.2.

Theorem 3.3 implies that $f_1 \in H^p$ for $0 < p < 1$. An examination of the argument for Theorem 3.3 shows that

$$M_p^p(r, f_1) \le A \| f \|_{F_1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|1 - re^{i\theta}|^p} \, d\theta$$

for $0 \leq r < 1$ and $0 < p < 1$, where A is a positive constant which does not depend on p. Since $| 1 - re^{i\theta} | \geq B | \theta |$ for $-\pi \leq \theta \leq \pi$ and $0 \leq r < 1$, where B is a constant, we obtain

$$M_p^p(r, f_1) \leq \frac{A \| f \|_{\mathcal{F}_1}}{2\pi B^p} \int_{-\pi}^{\pi} \frac{1}{|\theta|^p} \, d\theta = \frac{A \| f \|_{\mathcal{F}_1}}{B^p \pi^p (1-p)}.$$

There is a constant $C > 0$ such that $1/(1-p)^{1/p} \leq C/(1-p)$ for $0 < p < 1$. Therefore there is a constant D such that

$$\| f_1 \|_p \leq \frac{D}{1-p} \| f \|_{\mathcal{F}_1}^{1/p}$$

for $0 < p < 1$. It follows that

$$\overline{\lim_{p \to 1-}} \| f_1 \|_p \, (1-p) < \infty. \tag{10.5}$$

For $| z | < 1$ and $z \neq 0$ let $g(z) = f_2(1/z)$, and let $g(0) = 0$. Then

$$g(z) = -z \int_T \frac{\zeta}{1-\zeta z} \, d\mu(\zeta).$$

Hence there is a measure υ 0 $\mathcal{M}$ with $\| \upsilon \| = \| \mu \|$ and

$$g(z) = z \int_T \frac{1}{1-\overline{\zeta} z} \, d\upsilon(\zeta).$$

Thus g is of the form $z \cdot h$, where h 0 $\mathcal{F}_1$ and it follows that g 0 H^p for $0 < p < 1$. This yields f_2 0 $H^p(D')$ for $0 < p < 1$. The argument given previously for f_1 applies to g and yields

$$\overline{\lim_{p \to 1-}} \| f_2 \|_p \, (1-p) < \infty. \tag{10.6}$$

The limit

$$\lim_{r \to 1-} [f_1(re^{i\theta}) - f_2 (\frac{1}{r} e^{i\theta})] \tag{10.7}$$

exists for almost all θ 0 $[-\pi,\pi]$. We denote this limit by $F(\theta)$. We shall show that F is Lebesgue integrable on $[-\pi,\pi]$. If $|\zeta| = 1$, $0 < r < 1$ and $|z| = 1$ then

$$\frac{1}{1-\overline{\zeta}rz} - \frac{1}{1-\overline{\zeta}(z/r)} = \frac{1-r^2}{|1-\overline{\zeta}rz|^2} = \operatorname{Re}\left\{\frac{1+\overline{\zeta}rz}{1-\overline{\zeta}rz}\right\}.$$

Hence

$$f_1(re^{i\theta}) - f_2(\frac{1}{r}e^{i\theta}) = \int_T \operatorname{Re}\left\{\frac{1+\overline{\zeta}re^{i\theta}}{1-\overline{\zeta}re^{i\theta}}\right\} d\mu(\zeta).$$

To show that F 0 $L^1([-\pi,\pi])$ we may assume that μ 0 $\mathcal{M}^*$. Then

$$f_1(re^{i\theta}) - f_2(\frac{1}{r}e^{i\theta}) = \operatorname{Re} G(re^{i\theta}) \tag{10.8}$$

where

$$G(w) = \int_T \frac{1+\overline{\zeta}w}{1-\overline{\zeta}w}\, d\mu(\zeta) \qquad (|w|<1). \tag{10.9}$$

Let $u = \operatorname{Re} G$. The assumption μ 0 $\mathcal{M}^*$ implies that $u(w) > 0$ for $|w| < 1$. Also u is harmonic in $\mathcal{D}$, and hence

$$U(\theta) \equiv \lim_{r\to 1-} u(re^{i\theta})$$

exists for almost all θ 0 $[-\pi,\pi]$. We claim that U is Lebesgue integrable. To prove this, let $\{r_n\}$ be a sequence with $0 < r_n < 1$ for $n = 1,2,\ldots$ and $r_n \to 1$ as $n \to \infty$. For each n we define the function U_n by $U_n(\theta) = u(r_n e^{i\theta})$, $-\pi \le \theta \le \pi$. Then U_n is positive, continuous (and hence measurable), and $U_n(\theta) \to U(\theta)$ as $n \to \infty$ for almost all θ in $[-\pi,\pi]$. Fatou's lemma and the mean-value theorem yield

$$\int_{-\pi}^{\pi} U(\theta)\,d\theta \le \underline{\lim}_{n\to\infty} \int_{-\pi}^{\pi} U_n(\theta)\,d\theta = \underline{\lim}_{n\to\infty} \int_{-\pi}^{\pi} u(r_n e^{i\theta})\,d\theta = 2\pi u(0).$$

Since $U(\theta) \geq 0$ almost everywhere, this inequality shows that U is Lebesgue integrable on $[-\pi, \pi]$. Therefore F 0 L^1 $([-\pi, \pi])$. This completes the proof of the next theorem.

Theorem 10.2 *Let* f *be defined by* (10.1) *for* z 0 | \ T where μ 0 *M*, *and let* $f(\infty) = 0$. *Let* f_1 *and* f_2 *denote the restrictions of* f *to* D *and to* D′, *respectively. Then the following assertions hold.*

(a) f *is analytic in* $|^{\infty}$ \ T.

(b) f_1 0 H^p *and* f_2 0 H^p (D′) for $0 < p < 1$.

(c) $\overline{\lim_{p \to 1-}} \| f_1 \|_p \, (1-p) < \infty$ *and* $\overline{\lim_{p \to 1-}} \| f_2 \|_p \, (1-p) < \infty$.

(d) *The function* $F(\theta)$ *defined by* $F(\theta) = \lim_{r \to 1-} [f_1(re^{i\theta}) - f_2(\frac{1}{r} e^{i\theta})]$ *exists for almost all* θ *in* $[-\pi, \pi]$ *and* F 0 $L^1([-\pi, \pi])$.

Theorem 10.2 provides one direction of the characterization of the functions given by (10.1) for z 0 | \ T. The remainder of this chapter is devoted to proving the converse of Theorem 10.2, thus completing the characterization. The argument will show that the limit superior in the expressions in (c) can be replaced by the limit inferior.

Several lemmas are needed to complete the characterization. The first four lemmas deal with subharmonic functions. Lemma 10.3 and Lemma 10.4 can be found in Rado [1937; see p. 11 and p. 13].

Lemma 10.3 *Suppose that* u: Φ → | *is a subharmonic function in the domain* Φ *and let* Λ *be a domain with closure* $\overline{\Lambda} \subset \Phi$. *Then there is a sequence* $\{u_n\}$ $(n = 1, 2, \ldots)$ *of real-valued functions defined on* $\overline{\Lambda}$ *with the following properties.*

(a) u_n *is subharmonic in* Λ.

(b) *All second order derivatives of* u_n *exist and are continuous on* Λ.

(c) u_n *is nonincreasing on* $\overline{\Lambda}$.

(d) $u_n \to u$ on $\overline{\Lambda}$.

Lemma 10.4 *Let* Φ *be a domain, and suppose that all second order derivatives of the function* u: Φ → | *exist and are continuous on* Φ. *Then* u *is subharmonic in* Φ *if and only if* $\Delta u \geq 0$ *in* Φ, *where* Δ *denotes the laplacian.*

In the proof of the next lemma, we use a formula for the laplacian of the composition of a function with an analytic function. We briefly outline an argument for obtaining that formula. Let Ω and Φ be domains in $\mathbb{R}^2$. Suppose that F: $\Omega \to \Phi$ and G: $\Phi \to \mathbb{R}$ and let $H = G \circ F$. Assume that all second order partial derivatives of F and G exist on Ω and Φ, respectively. By composite differentiation ΔH, the laplacian of H, can be expressed in terms of the partial derivatives of F and G up to order 2. For (x, y) 0 Ω let F(x, y) = (u, v) and suppose that u + iv is an analytic function of z = x + iy. Then the Cauchy-Riemann equations, Δu = 0 and Δv = 0 yield a simplification of the general formula for ΔH. If in addition we use

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = |\, f'(z)\,|^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$$

we obtain the result

$$\Delta H = \left(\frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial v^2}\right) |\, f'(z)\,|^2 .$$

Thus $\Delta H = (\Delta G)\, |\, f'(z)\,|^2$.

Lemma 10.5 *Let Ω and Φ be domains in* |. *Suppose that* $f : \Omega \to \Phi$ *is analytic and* u: Φ → | *is subharmonic. Then the composition of* $u \circ f$ *is subharmonic on* Ω.

Proof: Let v = $u \circ f$. Since f is continuous on Ω and u is upper semicontinuous on Φ, v is upper semicontinuous on Ω. Hence it suffices to prove the mean value inequality for v. If *f* is constant, then v is constant and the mean value property holds. Thus we may assume *f* is not a constant function.

Let z_0 0 Ω. Suppose that r > 0 and {z: | $z – z_0$ | ≤ r} δ Ω. Choose r′ with r < r′ and {z: | $z – z_0$ | ≤ r′} δ Ω, and let

$$\Lambda = f\ (\{z : |\, z - z_0\,| < r'\}).$$

Then Λ is a domain and $\overline{\Lambda}$ δ Ω. Hence there is a sequence $\{u_n\}$ of subharmonic functions on Λ obeying the conclusions of Lemma 10.3. Lemma 10.4 implies that $\Delta u_n \geq 0$ in Λ. Let $v_n(z) = (u_n \circ f)(z)$ for n = 1, 2, … and for $|\, z - z_0\,| \leq r'$. Then $\Delta v_n = (\Delta u_n)|\, f'(z)\,|^2$ in $\{z : |\, z - z_0\,| < r'\}$. Since $\Delta u_n \geq 0$ in Λ this shows that $\Delta v_n \geq 0$ in $\{z : |\, z - z_0\,| < r'\}$. Lemma 10.4 implies that v_n is subharmonic in $\{z : |\, z - z_0\,| < r')$. In particular, this yields

$$v_n(z_0) \le \frac{1}{2\pi} \int_{-\pi}^{\pi} v_n(z_0 + re^{i\theta})\, d\theta \qquad (10.10)$$

for n = 1, 2, … .

By Lemma 10.3, $u_n \to u$ on $\overline{\Lambda}$. Therefore $v_n = u_n \circ f \to u \circ f = v$ on Λ. Also $\{v_n\}$ is nonincreasing and thus the monotone convergence theorem yields

$$\int_{-\pi}^{\pi} v_n(z_0 + re^{i\theta})\, d\theta \to \int_{-\pi}^{\pi} v(z_0 + re^{i\theta})\, d\theta.$$

Taking the limit on both sides of (10.10) yields

$$v(z_0) \le \frac{1}{2\pi} \int_{-\pi}^{\pi} v(z_0 + re^{i\theta})\, d\theta.$$

Next we define a function u_p: | → | for each $p > 0$. Lemma 10.6 will imply that u_p is subharmonic for certain values of p. The functions u_p will play a critical role in the proof of the converse to Theorem 10.2.

Let $p > 0$. We define $u_p(0) = 0$. If $z \ne 0$, $x = \text{Re } z$ and $y = \text{Im } z$, let

$$u_p(z) = |z|^p \cos(p\varphi) \qquad (10.11)$$

where

$$\varphi = \arctan\left(\frac{y}{|x|}\right) \qquad (10.12)$$

and $-\pi/2 \le \varphi \le \pi/2$.

Suppose that $z \ne 0$ and $\text{Re } z \ge 0$ and let $z = |z|e^{i\theta}$, where $-\pi/2 \le \theta \le \pi/2$. Then (10.12) gives $\varphi = \theta$. Hence

$$u_p(z) = \text{Re}\,(z^p) \qquad (10.13)$$

for $z \ne 0$, $\text{Re } z \ge 0$ and $-\pi/2 \le \arg z \le \pi/2$.

Next suppose that $\text{Re } z < 0$ and $\text{Im } z > 0$, and let $z = |z|e^{i\theta}$ where $\pi/2 < \theta < \pi$. Then $\varphi = \pi - \theta$ and hence

$$(-z)^p = (-|z|e^{i(\pi-\varphi)})^p = |z|^p\, e^{-ip\varphi}.$$

Therefore

$$\text{Re}\,[(-z)^p] = |z|^p \cos(p\varphi).$$

This shows that

$$u_p(z) = \text{Re}[(-z)^p] \tag{10.14}$$

for z with Re z < 0 and lm z > 0. In the case Re z < 0 and Im z $\leq$ 0, let $z = |z|e^{i\theta}$ where $-\pi \leq \theta < -\pi/2$. Then $\varphi = -\pi - \theta$, and (10.14) follows. Thus (10.14) holds when Re z < 0 and $-\pi/2 \leq \arg(-z) \leq \pi/2$.

The definition of u_p implies that u_p is continuous on $\mathbb{C}$. If Re z $\geq$ 0 and z $\neq$ 0, then $\varphi = \theta$ and

$$\lim_{p\to 1} u_p(z) = \lim_{p\to 1}\left[\,|z|^p \cos(p\varphi)\,\right] = |z|\cos\varphi = |z|\cos\theta = \text{Re}\ z.$$

If Re z < 0, then

$$\lim_{p\to 1} u_p(z) = |z|\cos\varphi = |z|\,(-\cos\theta) = -\text{Re}\ z.$$

Hence for all $z \in \mathbb{C}$,

$$\lim_{p\to 1} u_p(z) = |\text{Re}\ z|. \tag{10.15}$$

Lemma 10.6 *Let* p > 0 *and let* u_p *be the function defined by* (10.11) *and* (10.12) *where* $-\pi/2 \leq \varphi \leq \pi/2$. *If* $0 < p \leq 2$, *then* u_p *is subharmonic in* $\mathbb{C}$.

Proof: Since u_p is continuous on $\mathbb{C}$, it suffices to prove the mean value inequality at each point in $\mathbb{C}$.

Equation (10.13) shows that u_p is the real part of an analytic function in {z: Re z > 0}. Hence u_p is harmonic in {z: Re z > 0}, and u_p satisfies the mean value equality at each point z_0 where Re z_0 > 0. The same conclusion holds if Re z_0 < 0 because (10.14) implies that u_p is harmonic in {z: Re z < 0}.

It remains to prove the mean value inequality at points on the imaginary axis. Let z_0 = iy where $y \in \mathbb{R}$. First consider the case y = 0, and let r > 0. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u_p(re^{i\theta})\,d\theta = \frac{1}{2\pi}\left\{2\int_{-\pi/2}^{\pi/2} r^p \cos(p\theta)\,d\theta\right\}$$

$$= \frac{2r^p}{\pi p}\sin(p\pi/2).$$

The assumption $0 < p \le 2$ implies that $\sin(p\pi/2) \ge 0$. Therefore

$$u_p(0) = 0 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} u_p(re^{i\theta})\,d\theta.$$

Next consider the case $z_0 = iy$ where $y \in \mathbb{R}$ and $y \ne 0$. Let

$$\Omega = \mathbb{C} \setminus \{w \in \mathbb{R} : w \le 0\}$$

and define the function v by $v(z) = \text{Re}\,[z^p]$ where $z \in \Omega$ and $-\pi < \arg z < \pi$. Then v is harmonic in Ω. We claim that

$$v(z) \le u_p(z) \tag{10.16}$$

for $z \in \Omega$. If Re $z \ge 0$ and $z \ne 0$, then equality holds in (10.16) because of (10.13). Suppose that Re $z < 0$ and Im $z > 0$ and let $z = se^{i\theta}$ where $s > 0$ and $\pi/2 < \theta < \pi$. Hence

$$u_p(z) - v(z) = s^p\cos[p(\pi-\theta)] - s^p\cos(p\theta)$$

$$= 2\,s^p\sin[p(\theta-\pi/2)]\sin[p\pi/2].$$

Since $0 < p \le 2$ and $\pi/2 < \theta < \pi$, the previous expression is nonnegative. Thus (10.16) holds. Finally suppose Re $z < 0$ and Im $z < 0$. Then $z = se^{i\theta}$ where $s > 0$ and $-\pi < \theta < -\pi/2$. Since $0 < p \le 2$ this yields $\sin[p(\theta + \pi/2)] \le 0$. Hence

$$u_p(z) - v(z) = s^p\cos[p(-\pi-\theta)] - s^p\cos(p\theta)$$

$$= -2\,s^p\sin[p(\theta+\pi/2)]\sin[p\pi/2] \ge 0.$$

This completes the proof of (10.16).

Since v is harmonic in Ω and y is real with $y \ne 0$,

$$v(iy) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(iy + re^{i\theta})\, d\theta \qquad (10.17)$$

for $0 < r < | y |$. For $y \neq 0$, we have $u_p\,(iy) = v(iy)$. Hence (10.17) and (10.16) yield

$$u_p(iy) \le \frac{1}{2\pi} \int_{-\pi}^{\pi} u_p(iy + re^{i\theta})\, d\theta$$

for $0 < r < | y |$.

Lemma 10.7 *Suppose that* f 0 H^p *for* $0 < p < 1$ *and* $\varliminf_{p \to 1-} [\| f \|_p\ (1-p)] < \infty$. *Let* $U(\theta) = \text{Re}\,[\lim_{r\to 1-} f(re^{i\theta})]$. *Then* U *is defined almost everywhere in* $[-\pi, \pi]$. *If* U *is Lebesgue integrable on* $[-\pi, \pi]$, *then there is a real measure* μ 0 $\mathcal{M}$ *such that*

$$f(z) = \int_T \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta) + i\, \text{Im}\, f(0) \qquad (10.18)$$

for $| z | < 1$.

Proof: Since f 0 H^p for some $p > 0$, it follows that U is defined for almost all θ in $[-\pi, \pi]$. The assumption U 0 $L^1([-\pi, \pi])$ implies that g 0 $\mathcal{F}_1$, where g is defined for $| z | < 1$ by

$$g(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, U(\theta)\, d\theta. \qquad (10.19)$$

Hence Theorem 3.3 yields g 0 H^p for $0 < p < 1$. The function $u = \text{Re}\, g$ is the Poisson integral of the Lebesgue integrable function U. Therefore $\lim_{r\to 1-} u(re^{i\theta})$ exists and equals $U(\theta)$ for almost all θ, that is,

$$\text{Re}\left[\lim_{r\to 1-} g(re^{i\theta})\right] = \text{Re}\left[\lim_{r\to 1-} f(re^{i\theta})\right]$$

for almost all θ. Hence by considering the function $f - g$ it follows that in order to prove (10.18) we may assume that $\lim\limits_{r \to 1-} \operatorname{Re} f(re^{i\theta}) = 0$ for almost all θ.

Let u_p be the function defined by (10.12) and (10.13). Since $0 < p < 1$, Lemma 10.6 and Lemma 10.5 imply that $v_p \equiv u_p \circ f$ is subharmonic in $\mathbb{D}$. Let $F(\theta) = \lim\limits_{r \to 1-} f(re^{i\theta})$ for almost all θ in $[-\pi, \pi]$. Since u_p is continuous on $\mathbb{C}$, $V_p(\theta) \equiv \lim\limits_{r \to 1-} v_p(re^{i\theta})$ exists and equals $u_p(F(\theta))$ for almost all θ, and hence V_p is measurable. Equation (10.11) shows that $|u_p(w)| \le |w|^p$ for all $w \in \mathbb{C}$. Also $u_p(w) \ge 0$ for $0 < p < 1$. Thus

$$0 \le V_p(\theta) \le |F(\theta)|^p$$

for almost all θ. By hypothesis $f \in H^p$, and it follows that $F \in L^p([-\pi, \pi])$. Therefore $V_p \in L^1([-\pi, \pi])$.

We shall show that

$$\| V_p(\theta) - v_p(re^{i\theta}) \|_{L^1} \to 0 \tag{10.20}$$

as $r \to 1-$. As mentioned above, the assumptions on u_p and f imply that

$$v_p(re^{i\theta}) - V_p(\theta) \to 0 \tag{10.21}$$

as $r \to 1-$ for almost all θ. Also

$$\begin{aligned}\int_{-\pi}^{\pi} |v_p(re^{i\theta}) - V_p(\theta)|\, d\theta &\le \int_{-\pi}^{\pi} |v_p(re^{i\theta})|\, d\theta + \int_{-\pi}^{\pi} |V_p(\theta)|\, d\theta \\ &\le \int_{-\pi}^{\pi} |f(re^{i\theta})|^p\, d\theta + \int_{-\pi}^{\pi} |F(\theta)|^p\, d\theta \\ &\le 2\int_{-\pi}^{\pi} |F(\theta)|^p\, d\theta < \infty.\end{aligned}$$

Because of (10.21) the Lebesgue convergence theorem yields (10.20).

For $0 < r < 1$ and $|z| < 1$ let

$$S_p(z,r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right] v_p(re^{i\theta})\,d\theta. \tag{10.22}$$

Then $S_p(\cdot, r)$ is harmonic in D, and since v_p is continuous on $\{z: |z| = r\}$, $S_p(\cdot, r)$ extends continuously to $\overline{D}$ with $S_p(z,r) = v_p(rz)$ for $|z| = 1$. Since v_p is subharmonic the maximum principle yields

$$v_p(rz) \le S_p(z,r) \tag{10.23}$$

for $|z| < 1$ and $0 < r < 1$. The continuity of v_p and (10.23) imply that

$$v_p(z) \le \overline{\lim_{r\to 1-}}\, S_p(z,r). \tag{10.24}$$

Since

$$0 \le S_p(z,r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right]\left(v_p(re^{i\theta}) - V_p(\theta)\right)d\theta$$
$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right] V_p(\theta)\,d\theta$$

it follows that

$$\begin{aligned} S_p(z,r) \le \frac{1+|z|}{1-|z|}\frac{1}{2\pi}\int_{-\pi}^{\pi} |v_p(re^{i\theta}) - V_p(\theta)|\,d\theta \\ + \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right] V_p(\theta)\,d\theta. \end{aligned} \tag{10.25}$$

If Re w = 0 then $u_p(w) = |w|^p \cos(p\pi/2)$. Since $F(\theta) = 0$ almost everywhere this implies that $V_p(\theta) = |F(\theta)|^p \cos(p\pi/2)$ almost everywhere. Because of (10.20), the first integral in (10.25) tends to 0 as $r \to 1-$. Therefore

$$\overline{\lim_{r\to 1-}}\, S_p(z,r) \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right] |F(\theta)|^p \cos(p\pi/2)\,d\theta.$$

Hence (10.24) gives

$$v_p(z) \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right] |F(\theta)|^p \cos(p\pi/2)\, d\theta \tag{10.26}$$

for $|z|<1$.

There is a positive constant C such that $\cos(p\pi/2) \le C(1-p)$ for $0 < p < 1$. Hence the assumption

$$\varlimsup_{p\to 1-} \left[\| f \|_p (1-p)\right] < \infty$$

implies that the set of measures

$$\left\{\frac{1}{2\pi} |F(\theta)|^p \cos(p\pi/2)\, d\theta : 0 < p < 1\right\}$$

is bounded in the total variation norm. In particular, there is a sequence $\{p_n\}$ with $0 < p_n < 1$ and $p_n \to 1$ for which the associated sequence of measures is bounded. By the Banach–Alaoglu theorem this sequence has a weak* accumulation measure υ on $[-\pi,\pi]$. Equation (10.15) gives

$$\lim_{p\to 1-} v_p(z) = |\mathrm{Re}\ f(z)| \text{ for } |z|<1.$$

Hence by taking a weak* limit on a suitable subsequence of $\{p_n\}$, (10.26) yields

$$|\mathrm{Re}\ f(z)| \le \int_{-\pi}^{\pi} \mathrm{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right] d\upsilon(\theta) \tag{10.27}$$

for $|z|<1$.

Let $0 < r < 1$ and let $z = re^{i\varphi}$. Since $\upsilon \ge 0$, (10.27), Fubini's theorem and the mean value theorem for harmonic functions yield

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\operatorname{Re}\, f(re^{i\varphi})|\,d\varphi \le \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\int_{-\pi}^{\pi} \operatorname{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right] d\upsilon(\theta)\right\} d\varphi$$

$$= \int_{-\pi}^{\pi}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right] d\varphi\right\} d\upsilon(\theta)$$

$$= \int_{-\pi}^{\pi} d\upsilon(\theta) = \|\upsilon\|.$$

This implies that $\operatorname{Re} f \in h^1$. Therefore there is a real measure $\mu \in \mathcal{M}$ such that

$$\operatorname{Re}\, f(z) = \int_T \operatorname{Re}\left[\frac{\zeta+z}{\zeta-z}\right] d\mu(\zeta) \tag{10.28}$$

for $|z| < 1$.

Let the function f_0 be defined by

$$f_0(z) = \int_T \frac{\zeta+z}{\zeta-z}\, d\mu(\zeta) \tag{10.29}$$

for $|z| < 1$. Then f_0 is analytic in $\mathbb{D}$ and since μ is a real measure, $\operatorname{Re} f_0(z) = \operatorname{Re} f(z)$ for $|z| < 1$. Hence $f = f_0 + iy$ for some real number y. Since $f_0(0)$ is real, we have $y = \operatorname{Im} f(0)$. This proves (10.18).

We now prove the converse to Theorem 10.2.

Theorem 10.8 *Suppose that the function* $f : \mathbb{C}^\infty \setminus T \to \mathbb{C}$ *is analytic. Let* f_1 *and* f_2 *denote the restrictions of* f *to* $\mathbb{D}$ *and to* $\mathbb{C}^\infty \setminus \overline{\mathbb{D}}$, *respectively. Suppose that the following conditions hold.*

(a) $f(\infty) = 0$.

(b) $f_1 \in H^p$ *for* $0 < p < 1$ *and* $\varliminf\limits_{p\to 1-} [\,\|f_1\|_p (1-p)] < \infty$.

(c) $f_2 \in H^p(\mathbb{C}^\infty \setminus \overline{\mathbb{D}})$ *for* $0 < p < 1$ *and* $\varliminf\limits_{p\to 1-} [\,\|f_2\|_p (1-p)] < \infty$.

(d) *The function* F *defined almost everywhere on* $[-\pi,\pi]$ *by*

$$F(\theta) = \lim_{r\to 1-} \left[f_1(re^{i\theta}) - f_2\left(\frac{1}{r}e^{i\theta}\right)\right] \textit{ belongs to } L^1([-\pi,\pi]).$$

Then there exists μ 0 *M such that*

$$f(z) = \int_T \frac{1}{1-\overline{\zeta}z}\, d\mu(\zeta) \qquad (10.30)$$

for z 0 | \ T.

Proof: Suppose that f satisfies the assumptions given above. Let the function g_1 be defined by

$$g_1(z) = -i\left[f(z) + \overline{f(1/\overline{z})}\right] \qquad (10.31)$$

for z 0 $|^\infty$ \ T. Then g_1 is analytic in $|^\infty$ \ T. The assumptions (b) and (c) imply that g_1 0 H^p for $0 < p < 1$ and

$$\varliminf_{p\to 1-} [\|g_1\|_p\ (1-p)] < \infty.$$

There is a set E δ [−π, π] with measure 2π such that

$$F_1(\theta) \equiv \lim_{r\to 1-} f_1(re^{i\theta}),\ F_2(\theta) \equiv \lim_{r\to 1-} f_2(\frac{1}{r}e^{i\theta})$$

and

$$G_1(\theta) \equiv \lim_{r\to 1-} g_1(re^{i\theta})$$

exist for all θ in E. For almost all θ,

$$G_1(\theta) = -i[F_1(\theta) + \overline{F_2(\theta)}]$$

and hence Re $G_1(\theta)$ = Im $F_1(\theta)$ – Im $F_2(\theta)$ = Im $F(\theta)$, where F is defined in (d). Since F 0 L^1 ([−π, π]), it follows that Re G_1 0 L^1 ([−π, π]). Lemma 10.7 yields a real measure υ_1 0 *M* such that

$$g_1(z) = \int_T \frac{\zeta+z}{\zeta-z}\, d\upsilon_1(\zeta) + i\ \mathrm{Im}\, g_1(0) \qquad (10.32)$$

for $| z | < 1$. The relation (10.31) implies that

$$\overline{g_1(1/\overline{z})} = -g_1(z) \qquad (10.33)$$

for $z\ 0\ |^{\infty} \setminus T$. Assume that $z\ 0\ |^{\infty} \setminus \overline{D}$. Since (10.32) holds at $1/\overline{z}$ and since υ_1 is a real measure, (10.33) yields

$$g_1(z) = -\overline{g_1(1/\overline{z})} = -\int_T \frac{\overline{\zeta} + \frac{1}{z}}{\overline{\zeta} - \frac{1}{z}}\, d\upsilon_1(\zeta) + i \operatorname{Im} g_1(0)$$

$$= \int_T \frac{\zeta + z}{\zeta - z}\, d\upsilon_1(\zeta) + i \operatorname{Im} g_1(0).$$

Thus (10.32) holds for all $z\ 0\ | \setminus T$.

The argument will continue with a similar analysis of the function g_2 defined by

$$g_2(z) = f(z) - \overline{f(1/\overline{z})} \qquad (z\ 0\ |^{\infty} \setminus T). \qquad (10.34)$$

Note that g_2 is analytic in $|^{\infty} \setminus T$, $g_2\ 0\ H^p$ $(0 < p < 1)$, and

$$\varliminf_{p \to 1-} \left[\| g_2 \|_p \ (1 - p) \right] < \infty.$$

The function

$$G_2(\theta) \equiv \lim_{r \to 1-} g_2(re^{i\theta})$$

exists for almost all θ, and $G_2(\theta) = F_1(\theta) - \overline{F_2(\theta)}$. It follows as in the previous argument that $\operatorname{Re} G_2(\theta) = \operatorname{Re} F(\theta)$ for almost all θ. Thus assumption (d) yields $\operatorname{Re} G_2\ 0\ L^1([-\pi, \pi])$. By Lemma 10.7, there is a real measure $\upsilon_2\ 0\ \mathcal{M}$ such that

$$g_2(z) = \int_T \frac{\zeta + z}{\zeta - z}\, d\upsilon_2(\zeta) + i \operatorname{Im} g_2(0) \qquad (10.35)$$

for $|z| < 1$. The argument given for g_1 applies here and shows that (10.35) holds for $z \in \mathbb{C} \setminus T$.

The equations (10.31) and (10.34) give

$$f(z) = \frac{1}{2}\left[i\, g_1(z) + g_2(z)\right] \tag{10.36}$$

for $z \in \mathbb{C}^{\infty} \setminus T$. Thus (10.32) and (10.35) yield

$$f(z) = \int_T \frac{\zeta + z}{\zeta - z}\, d\upsilon(\zeta) + b \tag{10.37}$$

for $z \in \mathbb{C} \setminus T$, where $\upsilon = \frac{1}{2}(i\upsilon_1 + \upsilon_2)$ and

$$b = \frac{1}{2}\left[-\operatorname{Im} g_1(0) + i \operatorname{Im} g_2(0)\right] = \frac{1}{2} f(0).$$

Equation (10.37) can be written as

$$f(z) = \int_T \frac{1}{1 - \overline{\zeta} z}\, d\mu(\zeta) \tag{10.38}$$

where $\mu = 2\upsilon + \lambda$ and λ is a multiple of Lebesgue measure on T. This proves (10.30). □

Theorems 10.2 and 10.8 give an intrinsic analytic description of the functions that can be represented by a Cauchy transform on $\mathbb{C} \setminus T$. There is no comparable characterization of $\mathcal{F}_1$, or more generally, of $\mathcal{F}_\alpha$ for $\alpha > 0$. The characterization of $\mathcal{F}_1$ given in Theorem 2.21 is implicit. The formula (1.1) which defines $\mathcal{F}_\alpha$ for $\alpha > 0$ actually defines a function which is analytic in $\mathbb{C} \setminus T$ when α is a positive integer. It is an open problem to find results comparable to Theorems 10.2 and 10.8 for $\alpha = 2, 3, \ldots$.

NOTES

The Hardy spaces $H^p(\Omega)$ are defined for various domains $\Omega \subset \mathbb{C}$. A reference for the definitions and results is Duren [1970; see Chapter 10]. Theorem 2 is a result of V.I. Smirnov. The result in Lemma 5, namely, that the composition of a subharmonic function with an analytic function is subharmonic, is a classical fact. The argument given here is in Ransford [1995; p. 50]. The introduction of the function u_p and the proof of Lemma 6 are due to Pichorides [1972], where other applications of that lemma are given. The proof given here for Theorem 8 is contained in Aleksandrov [1981; see p. 22-25].